

POSITIVE SPEED SELF-AVOIDING WALKS ON GRAPHS WITH MORE THAN ONE END

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ABSTRACT. A self-avoiding walk (SAW) is a path on a graph that visits each vertex at most once. The mean square displacement of an n -step SAW is the expected value of the square of the distance between the ending point and the starting point of an n -step SAW, where the expectation is taken with respect to the uniform measure on n -step SAWs starting from a fixed vertex. It is conjectured that the mean square displacement of an n -step SAW is asymptotically $n^{2\nu}$, where ν is a constant. Computing the exact values of the exponent ν on various graphs has been a challenging problem in mathematical and scientific research for long.

In this paper we show that on any locally finite Cayley graph of an infinite, finitely-generated group with more than two ends, the number of SAWs whose end-to-end distances are linear in lengths has the same exponential growth rate as the number of all the SAWs. We also prove that for any infinite, finitely-generated group with more than one end, there exists a locally finite Cayley graph on which SAWs have positive speed - this implies that the mean square displacement exponent $\nu = 1$ on such graphs.

These results are obtained by proving more general theorems for SAWs on quasi-transitive graphs with more than one end, which make use of a variation of Kesten's pattern theorem in a surprising way, as well as the Stallings's splitting theorem. Applications include proving that SAWs have positive speed on the square grid in an infinite cylinder, and on the infinite free product graph of two connected, quasi-transitive graphs.

1. INTRODUCTION

Self-avoiding walks, which are paths on graphs visiting no vertex more than once, were first introduced as a model for long-chain polymers in chemistry ([7], see also [21]). Despite its simple definition, SAWs have been notoriously difficult to enumerate and to study the geometric properties due to the fact that SAWs are, in general, non-Markovian.

In this paper, we consider SAWs on quasi-transitive graphs. Let $G = (V, E)$ be an infinite, connected graph, and let $\text{Aut}(G)$ be the automorphism group for G . We say that G is **quasi-transitive**, if there exists a subgroup Γ of $\text{Aut}(G)$ acting quasi-transitively on G , i.e. there exists a finite set of vertices $W \subset V$, $|W| < \infty$, such

that for any $v \in V$, there exist $w \in W$ and $\gamma \in \Gamma$ with $w = \gamma v$. The set W is called a **fundamental domain**.

The **connective constant** for SAWs on a quasi-transitive graph is the exponential growth rate of the number of n -step SAWs starting from a fixed vertex. More precisely, let $c_n(v)$ be the number of n -step SAWs starting from a fixed vertex $v \in V$, the connective constant μ is defined to be

$$\boxed{\text{cc}} \quad (1.1) \quad \mu := \lim_{n \rightarrow \infty} [\sup_{v \in V} c_n(v)]^{\frac{1}{n}}$$

The limit on the right hand side of (1.1) is known to exist by a sub-additivity argument. It is proved in [17] that the connective constant μ defined in (1.1), can be expressed as follows

$$\boxed{\text{cc1}} \quad (1.2) \quad \mu = \lim_{n \rightarrow \infty} c_n(v)^{\frac{1}{n}}, \quad \forall v \in V.$$

Although the definition of the SAW is quite simple, a lot of fundamental questions concerning SAWs remain unknown. For example, it is still an open problem to compute the exact value of the connective constant for the 2-dimensional square grid. A recent breakthrough is a proof of the fact that the connective constant of the hexagonal lattice is $\sqrt{2 + \sqrt{2}}$; see [5]. See [14, 13, 16, 10, 11] for results concerning bounds of connective constants on quasi-transitive graphs; and see [12, 15] for results concerning the dependence of connective constants on local structures of graphs.

The number of **ends** of a connected graph is the supremum over its finite subgraphs of the number of infinite components that remain after removing the subgraph.

Let $G = (V, E)$ be an infinite, connected, quasi-transitive graph with more than one end. Let $\Gamma \subseteq \text{Aut}(G)$ be a subgroup of the automorphism group of G acting quasi-transitively on G . We may make the following assumptions on the graph G .

$\boxed{\text{ap31}}$ **Assumption 1.1.** *There exist a finite set of vertices S , $S \subset V$ and $|S| < \infty$, such that*

- (1) S is connected;
- (2) $G \setminus S$ (the graph obtained from G by removing all the vertices in S and their incident edges) has at least two infinite components;
- (3) for any component A of $G \setminus S$, let $\partial_A S$ be the set consisting of all the vertices in S incident to a vertex in A . There exists an infinite component B of $G \setminus S$ and a graph automorphism $\gamma \in \Gamma$, such that $\gamma A \subseteq B$; for any $v \in \partial_A S$ $\gamma v \in \partial_B S \cup B$, v and γv are joined by a path in $G \setminus (A \cup \gamma A)$, whose length is bounded above by a constant N independent of A, v . Denote γ by $\phi(S, A) := \gamma$.

$\boxed{\text{ap32}}$ **Assumption 1.2.** *There exist a finite set of vertices S , $S \subset V$ and $|S| < \infty$ satisfying Assumption 1.1. Moreover, assume that*

- there exist a finite set of vertices S' , such that $S \subseteq S'$. Let $\partial S'$ be the set consisting of all the vertices in S' incident to a vertex in $G \setminus S'$. For any two distinct vertices $u, v \in \partial S'$, there exists an SAW l_{uv} joining u and v and visiting every vertex in S .

Here are the main results of the paper.

m31 **Theorem 1.3.** *Let $G = (V, E)$ be an infinite, connected, quasi-transitive graph with more than one end. Let μ be the connective constant of G . Let π_n^v be an n -step SAW on G starting from a fixed vertex v , and let*

$$\|\pi_n^v\| = \text{dist}_G(\pi(n), \pi(0)),$$

where $\text{dist}_G(\cdot, \cdot)$ is the graph distance on G .

A. If G satisfies Assumption 1.1, then there exists $a \in (0, 1]$

$$\limsup_{n \rightarrow \infty} \sup_{v \in V} |\{\pi_n^v : \|\pi_n^v\| \geq an\}|^{\frac{1}{n}} = \mu.$$

B. If G satisfies Assumption 1.2, then π_n^v has positive speed, i.e., there exist constants $C, \alpha, \beta > 0$, such that

$$\mathbb{P}_n(\|\pi_n^v\| \leq \alpha n) \leq Ce^{-n\beta}.$$

where \mathbb{P}_n is the uniform measure on the set of n -step SAWs on G starting from a fixed vertex.

The approach to prove Theorem 1.3 is to consider a finite “cut set” S , such that SAWs, once crossing this “cut set”, will move to another component of $G \setminus S$ and most of them may never come back again. The analysis involves arguments and technical details inspired by the pattern theorem ([19]), see also ([21, 4, 13]). Applications of Theorem 1.3 include a proof that SAWs on an infinite cylindrical square grid have positive speed, and that SAWs on an infinite free product graph of two quasi-transitive, connected graphs have positive speed.

Example 1.4. (Cylinder) Consider the quotient graph of the square grid \mathbb{Z}^2 , $\mathbb{Z} \times \mathbb{Z}_l$, where l is a positive integer. This is a graph with two ends. We can choose $S = \{0\} \times \mathbb{Z}_l$ and $S' = \{-1, 0, 1\} \times \mathbb{Z}$. Then Assumption 1.2 is satisfied and SAWs have positive speed. See also [8] for discussions about SAWs on a cylinder.

df41 **Definition 1.5. (Free product of graphs)** Let $G_1 = (V_1, E_1, o_1)$, $G_2 = (V_2, E_2, o_2)$ be two connected and quasi-transitive, rooted graphs with vertex sets V_1, V_2 ; edge sets E_1, E_2 and roots $o_1 \in V_1, o_2 \in V_2$, respectively. Assume that $|V_i| \geq 2$, for $i = 1, 2$. Let $V_i^\times = V_i \setminus \{o_i\}$ for $i = 1, 2$; and let $I(x) = i$ if $x \in V_i^\times$. Define

$$V := V_1 * V_2 = \{x_1 x_2 \dots x_n \mid n \in \mathbb{N}, x_k \in V_1^\times \cup V_2^\times, I(x_k) \neq I(x_{k+1})\} \cup \{o\}$$

We define an edge set E for the vertex set V as follows: if $i \in \{1, 2\}$ and $x, y \in V_i$, and $(x, y) \in E_i$, then $(wx, wy) \in E$ for all $w \in V$. See [9] for discussions of SAWs on free product graphs of quasi-transitive graphs.

tm51 **Theorem 1.6.** *Let $G = (V, E)$ be the free product graph of two connected and quasi-transitive, rooted graphs $G_1 = (V_1, E_1, o_1)$ and $G_2 = (V_2, E_2, o_2)$ with $|V_i| \geq 2$, for $i = 1, 2$, as defined in 1.5. Then SAWs on G have positive speed.*

Theorem 1.3 also has the following corollaries.

mg **Theorem 1.7.** *Let Γ be an infinite, finitely-generated group with more than two ends. Let $G = (V, E)$ be a locally finite Cayley graph of Γ . For $v \in V$ Let π_n^v be an n -step SAW on G starting from π . Then*

$$\limsup_{n \rightarrow \infty} |\{\pi_n^v : \|\pi_n^v\| \geq an\}|^{\frac{1}{n}} = \mu.$$

t15 **Theorem 1.8.** *Let Γ be an infinite, finitely-generated group with more than one end. There exists a locally finite Cayley graph $G = (V, E)$ of Γ , such that SAWs on G have positive speed.*

For a graph satisfying Assumption 1.2, Theorem 1.3 implies that the mean square displacement of SAWs on the graph is of the order n^2 , i.e.

$$\langle \|\pi\|^2 \rangle_n \sim n^2,$$

where $\langle \cdot \rangle_n$ is the expectation taken with respect to the uniform measure for n -step SAWs on G starting from a fixed vertex; and “ \sim ” means that there exist constants $C_1, C_2 > 0$, independent of n , such that $C_1 n^2 \leq \langle \|\pi\|^2 \rangle_n \leq C_2 n^2$.

The mean square displacement exponent ν for SAWs, defined by

$$\langle \|\pi\|^2 \rangle_n \sim n^{2\nu},$$

has been an interesting topic to mathematicians and scientists for long. It is conjectured that $\nu = \frac{3}{4}$ for SAWs on \mathbb{Z}^2 , $\nu = 0.5$ for SAWs on \mathbb{Z}^d with $d \geq 4$, and that $\nu = 1$ for SAWs on a non-amenable graph with bounded vertex degree.

The conjecture that $\nu = 1$ when $d \geq 5$ was proved in [3, 18]. See [1] for related results when $d = 4$, and [4] for related results for $d \geq 2$.

It is proved in [24] that if a non-amenable Cayley graph satisfies

cd12 (1.3) $(d - 1)\rho\mu^{-1} < 1,$

then SAWs have positive speed. Here d is the vertex degree, ρ is the spectral radius for the transition matrix of the simple random walk on the graph, and μ is the connective constant as defined in (1.1). Combining with the results in [23, 2, 25], it is known that for any finitely generated non-amenable group, there exists a locally finite Cayley graph on which SAWs have positive speed.

It is proved in [22] that SAWs have positive speed for certain regular tilings of the hyperbolic plane. An upper bound of the spectral radius for a planar graph with given maximal degree is proved in [6], which, combining with (1.3), can be used to show that SAWs have positive speed on a large class of planar graphs.

The organization of the paper is as follows. In Section 2, we prove Theorem 1.3 A. In Section 3, we prove Theorem 1.3 B. Theorems 1.7 and 1.8 are proved in Section 4. In Section 5, we prove Theorem 1.6.

2. PROOF OF THEOREM 1.3 A.

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This section is devoted to prove Theorem 1.3 A.

Let $G = (V, E)$ be a graph satisfying the assumption of Theorem 1.3. Let S be a finite set of vertices satisfying Assumption 1.1. Recall that Γ is a subset of $\text{Aut}(G)$ acting quasi-transitively on G . Let ΓS be the set of images of S under Γ . By quasi-transitivity of G , for each $\gamma \in \Gamma$, γS still satisfies Assumption 1.1.

Let π be an n -step SAW on G . We say that E^* occurs at the j th step of π if there exists $\gamma S \in \Gamma S$ such that $\pi(j) \in \gamma S$, and all the vertices of γS are visited by π . For $k \geq 1$, we say that E_k occurs at the j th step of π , if there exists $\gamma S \in \Gamma S$, such that $\pi(j) \in \gamma S$, and at least k vertices of γS are visited by π . We say that \tilde{E}_k occurs at the j th step of π if E^* or E_k (or both) occur there.

In the following, we will use E to denote any of E^* , E_k or \tilde{E}_k . If m is a positive integer, we say that $E(m)$ occurs at the j th step of π if E occurs at the m th step of the $2m$ -step subwalk $(\pi(j-m), \dots, \pi(j+m))$. (If $j-m < 0$ or $j+m > n$, then an obvious modification must be made in this definition: for $j-m < 0$, it means that E occurs at the j th step of $(\pi(0), \dots, \pi(j+m))$; for $j+m > n$, it means that E occurs at the m th step of $(\pi(j-m), \dots, \pi(n))$). In particular, if $E(m)$ occurs at the j th step of π , then E occurs at the j th step of π .

Let $c_n(v)$ be the number of n -step SAWs on G starting from a fixed vertex v . For $r \geq 0$, let $c_n^v(r, E)$ (resp. $c_n^v(r, E(m))$) be the number of n -step SAWs starting from v for which E (resp. $E(m)$) occurs at no more than r different steps.

Let

$$\begin{aligned} c_n &= \sup_{v \in V} c_n(v); \\ c_n(r, E) &= \sup_{v \in V} c_n^v(r, E); \\ \lambda(E) &= \limsup_{n \rightarrow \infty} c_n(0, E)^{\frac{1}{n}}. \end{aligned}$$

Let μ be the connective constant of G as defined in (1.1). We have that $\lambda(E) < \mu$ if and only if

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(2.1) there exist $\epsilon > 0$, $M \in \mathbb{N}$, such that $c_m(0, E) < [\mu(1 - \epsilon)]^m$, for $m \geq M$.

11 **Lemma 2.1.** *Suppose that*

1k (2.2) $\lambda(E) < \mu.$

Let ϵ, M satisfy (2.1), and let $m \geq M$ satisfy

em2 (2.3) $c_m \leq [\mu(1 + \epsilon)]^m.$

Then there exist $a = a(\epsilon, m)$ and $R = R(\epsilon, m) \in (0, 1)$, such that

$$\limsup_{n \rightarrow \infty} c_n(an, E(m))^{\frac{1}{n}} < R\mu.$$

Proof. Assume that (2.2) holds. Let ϵ, m satisfy (2.3). Since $c_m(0, E) = c_m(0, E(m))$, we have

$$c_m(0, E(m)) < [\mu(1 - \epsilon)]^m.$$

Let π be an n -step SAW on G and $N = \lfloor \frac{n}{m} \rfloor$. If $E(m)$ occurs in no more than r steps of π , then $E(m)$ occurs at no more than r of the m -step subwalks

$$(\pi((j-1)m), \pi((j-1)m+1), \dots, \pi(jm)), \quad 1 \leq j \leq N.$$

Counting the number of ways in which r or fewer of these subwalks can contain an occurrence of $E(m)$, we have that

$$\begin{aligned} c_n(r, E(m)) &\leq \sum_{i=1}^r \binom{N}{i} c_m^i [c_m(0, E(m))]^{N-i} c_{n-Nm} \\ &\leq \mu^{Nm} c_{n-Nm} \sum_{i=1}^r \binom{N}{i} (1 + \epsilon)^{im} (1 - \epsilon)^{(N-i)m} \end{aligned}$$

For ξ small and positive, we have

$$\begin{aligned} &\sum_{i=0}^{\xi N} \binom{N}{i} (1 + \epsilon)^{im} (1 - \epsilon)^{(N-i)m} \\ &\leq (\xi N + 1) \binom{N}{\xi N} \left(\frac{1 + \epsilon}{1 - \epsilon} \right)^{\xi Nm} (1 - \epsilon)^{Nm} \end{aligned}$$

The N th root of the right hand side converges as $N \rightarrow \infty$ to

$$f(\xi) = \frac{1}{\xi^\xi (1 - \xi)^{1-\xi}} \left(\frac{1 + \epsilon}{1 - \epsilon} \right)^{\xi m} (1 - \epsilon)^m,$$

which is strictly less than 1 for $0 < \xi < \xi_0$, and some $\xi_0 = \xi_0(\epsilon, m) > 0$. Therefore when $0 < a < \frac{\xi}{m}$, and $R = f(\xi)^{\frac{1}{m}}$, we have

$$c_n(an, E(m))^{\frac{1}{n}} < R\mu,$$

and the proof is complete. \square

qtc **Lemma 2.2.** *Let $G_1 = (V(G_1), E(G_1))$ be a component of $G \setminus \Gamma S$. There exists a subgroup Γ_1 of Γ acting quasi-transitively on G_1 .*

Proof. Since Γ acts on G quasi-transitively, V has finitely many orbits under the action of Γ . Let W_1 be the subset of $V(G_1)$ consisting of one representative in each orbit of V under the action of Γ , such that the intersection of the orbit with $V(G_1)$ is nonempty, then $|W_1| < \infty$.

Let

$$\Gamma_1 = \{\gamma \in \Gamma : \forall w \in W_1, \gamma w \in V(G_1)\}.$$

Then it is straightforward to check that Γ_1 is a subgroup of Γ , and that Γ_1 acts on G_1 quasi-transitively. \square

lcc **Lemma 2.3.** *There exists a component $G_1 = (V(G_1), E(G_1))$ of $G \setminus \Gamma S$, such that $\lambda(\tilde{E}_1)$ is the connective constant of G_1 , i.e.*

$$\text{d1} \quad (2.4) \quad \lambda(\tilde{E}_1) = \lim_{n \rightarrow \infty} \sup_{v \in V(G_1)} \tilde{c}_n(v)^{\frac{1}{n}}$$

$$\text{d2} \quad (2.5) \quad = \lim_{n \rightarrow \infty} \tilde{c}_n(v)^{\frac{1}{n}}, \quad \forall v \in V(G_1)$$

where $\tilde{c}_n(v)$ is the number of n -step SAWs on G_1 starting from v .

Proof. By definition of $\lambda(\tilde{E}_1)$, we have

$$\lambda(\tilde{E}_1) = \lim_{n \rightarrow \infty} \sup_{v \in V \setminus \Gamma S} \bar{c}_n(v)^{\frac{1}{n}},$$

where $\bar{c}_n(v)$ is the number of n -step SAWs starting from v on the component of $G \setminus \Gamma S$ including v .

By the quasi-transitivity of $G \setminus \Gamma S$ under the action of Γ , there exists $v_1 \in G \setminus \Gamma S$, such that

$$\lambda(\tilde{E}_1) = \limsup_{n \rightarrow \infty} \bar{c}_n(v_1)^{\frac{1}{n}}.$$

Let G_1 be the component of $G \setminus \Gamma S$ containing v_1 , then

$$\lambda(\tilde{E}_1) = \limsup_{n \rightarrow \infty} \bar{c}_n(v_1)^{\frac{1}{n}} \leq \lim_{n \rightarrow \infty} \sup_{v \in V(G_1)} \bar{c}_n(v)^{\frac{1}{n}} \leq \lim_{n \rightarrow \infty} \sup_{v \in V \setminus \Gamma S} \bar{c}_n(v)^{\frac{1}{n}} = \lambda(\tilde{E}_1).$$

Therefore $\lambda(\tilde{E}_1)$ is the connective constant for G_1 , and (2.4) follows. The identity (2.5) follows from Lemma 2.2 and (1.1), (1.2). \square

12 **Lemma 2.4.** $\lambda(\tilde{E}_1) < \mu$.

Proof. Since Γ acts on G quasi-transitively, let W be a fundamental domain such that $|W| < \infty$. Let

$$N_0 = \max_{w_1, w_2 \in W} \text{dist}_G(w_1, w_2).$$

For any $v \in V$, let

$$B_G(v, N_0) = \{u \in V : \text{dist}_G(v, u) \leq N_0\}.$$

It is not hard to see that for any $v \in V$, $B_G(v, N_0)$ contains a vertex in each orbit of V under the action of Γ . Therefore for any $v \in V$, there exists $\gamma S \in \Gamma S$, such that

$$B_G(v, N_0) \cap \gamma S \neq \emptyset.$$

Let G_1 be a component of $G \setminus \Gamma S$, such that $\lambda(\tilde{E}_1)$ is the connective constant of G_1 . The existence of G_1 is guaranteed by Lemma 2.3. Let $T_n(v)$ be the set of all n -step SAWs on G_1 starting from the vertex v . For each $\pi \in T_n(v)$, find indices j_1, \dots, j_u , such that for $i \neq l$, we have

$$\boxed{\text{a}} \quad (2.6) \quad \text{dist}_{G_1}(\pi(j_i), \pi(j_l)) \geq 8(N_0 + |S|) + N,$$

here N is given by Assumption 1.1 (3).

We may assume that $u = \kappa n$. For each $B_G(\pi(j_i), N_0)$, there exists $\gamma_{j_i} S \in \Gamma S$, such that

$$B_G(\pi(j_i), N_0) \cap \gamma_{j_i} S \neq \emptyset.$$

Let $\pi(t_i)$ be a closest vertex on π , in graph distance of G , to $\gamma_{j_i} S$. Then

$$\boxed{\text{b}} \quad (2.7) \quad \text{dist}_G(\pi(j_i), \pi(t_i)) \leq 2N_0 + |S|.$$

By rearrangements if necessary, we may assume that

$$t_1 < t_2 < \dots < t_{\kappa n}.$$

We choose a subset of indices $H \subset \{t_1, t_2, \dots, t_{\kappa n}\}$ to perform a manipulation, which will be described later. Assume that $H = \{h_1, \dots, h_{\delta n}\}$, where $0 < \delta < \kappa$, and $h_a < h_b$ if $a < b$. For each $h_l \in H$ with $h_l = t_i$, let $S_l = \gamma_{j_i} S$.

We construct a new SAW π_1 as follows

- let α_1 be the subwalk of π from $\pi(0)$ to $\pi(h_1)$;
- use a shortest path to join $\pi(h_1)$ and S_1 , denoted by ω_1 ;
- let A_1 be the component of $G \setminus S_1$ containing $\pi(h_1)$; use $\phi(S_1, A_1)$ to map the concatenation of the reversed ω_1 and $\pi(h_1), \pi(h_1+1), \dots, \pi(n)$ to another component of $G \setminus S_1$, and denote the image of the concatenation of the two subwalks under $\phi(S_1, A_1)$ by θ_1 - Here $\phi(S_1, A_1)$ is given as in Assumption 1.1 (3);

- use an SAW β_1 in $G \setminus [A_1 \cup \phi(S_1, A_1)A_1]$ to join the last vertex of ω_1 and its image under $\phi(S_1, A_1)$ (note that the existence of β_1 is guaranteed by Assumption 1.1);
- let π_1 be the concatenation of α_1 , ω_1 , β_1 and θ_1 .

It is not hard to check that π_1 is indeed an SAW. Note that α_1 is a subwalk of both π and π_1 ; and the subwalk $\pi \setminus \alpha_1$ is mapped, by $\phi(S_1, A_1)$ to a subwalk of θ_1 . Let $h_1^1 = h_1$, and let h_i^1 ($2 \leq i \leq \delta n$) be the step in π_1 of the image of $\pi(h_i)$ under $\phi(S_1, A_1)$.

We claim that for $2 \leq i \leq \delta n$, $\pi_1(h_i^1)$ is a closest vertex on π_1 to $\phi(S_1, A_1)S_i$. To see why that is true, first of all, since $\pi_1(h_2^1 - h_2 + h_1), \dots, \pi_1(n + h_2^1 - h_2)$ is the image of $\pi(h_1), \dots, \pi(n)$ under $\phi(S_1, A_1)$, we have that for $2 \leq i \leq \delta n$, $\pi_1(h_i^1)$ is a closest vertex on the subwalk $\pi_1(h_1^1 - h_2 + h_1), \dots, \pi_1(n - h_2 + h_2^1)$ to $\phi(S_1, A_1)S_i$. Moreover, we have

$$\text{dist}_G(\pi_1(h_i^1), \phi(S_1, A_1)S_i) \leq N_0, \text{ for } 2 \leq i \leq \delta n.$$

However, for any vertex v on the subwalk $\pi(0), \dots, \pi_1(h_2^1 - h_2 + h_1)$, we have

$$\boxed{\text{c}} \quad (2.8) \quad \text{dist}_G(v, \phi(S_1, A_1)S_i) > N_0, \text{ for } 2 \leq i \leq \delta n.$$

To see why (2.8) is true, note that from (2.6), (2.7), we have

$$\text{dist}_G(\pi_1(h_2^1 - h_2 + h_1), \phi(S_1, A_1)S_i) \geq 3N_0 + 6|S| + N.$$

Then for any $u \in \beta_1 \cup [\theta_1 \setminus (\pi_1(h_2^1 - h_2 + h_1), \dots, \pi_1(n - h_2 + h_2^1))] \cup \alpha_1 \cup \omega_1$, we have

$$\text{dist}_G(\pi_1(h_2^1 - h_2 + h_1), \phi(S_1, A_1)S_i) \geq 2N_0 + 5|S|.$$

This way we obtain (2.8), and the claim follows.

Let

$$\pi_0 := \pi, \quad h_i^0 = h_i,$$

for $1 \leq i \leq \delta n$. Assume that we have constructed SAWs π_1, \dots, π_s , where $1 \leq s < \delta n$. Assume that $\alpha_s = (\pi_{s-1}(0), \dots, \pi_{s-1}(h_s^{s-1}))$ is a subwalk of both π_{s-1} and π_s , and that $\pi_{s-1} \setminus \alpha_s$ is mapped by a graph automorphism $\phi_s \in \Gamma$ to a subwalk of π_s . Let $h_i^s = h_i^{s-1}$, for $1 \leq i \leq s$, and let h_i^s ($s+1 \leq i \leq \delta n$) be the step in π_s of the image of $\pi_{s-1}(h_i^{s-1})$ under the graph isomorphism ϕ_s . Assume also that $\pi_s(h_q^s)$ is the closest vertex on π_s to $\phi_s \dots \phi_1 S_q$ for $s+1 \leq q \leq \delta n$.

Now we construct an SAW π_{s+1} , following the procedure below.

- let α_{s+1} be the subwalk of π_s from $\pi_s(0)$ to $\pi_s(h_{s+1}^s)$;
- use a shortest path ω_{s+1} to join $\pi_s(h_{s+1}^s)$ and $\tilde{S}_{s+1} := \phi_s \dots \phi_1 S_{s+1}$;
- let A_{s+1} be the component of $G \setminus \tilde{S}_{s+1}$ containing $\pi_s(h_{s+1}^s)$; use $\phi_{s+1} := \phi(\tilde{S}_{s+1}, A_{s+1})$ to map the concatenation of the reversed ω_{s+1} and $\pi_s(h_{s+1}^s), \dots, \pi_s(n +$

- use an SAW β_{s+1} in $G \setminus [A_{s+1} \cup \phi(\tilde{S}_{s+1}, A_{s+1})A_{s+1}]$ to join the last vertex of ω_{s+1} and its image under ϕ_{s+1} (note that the existence of β_{s+1} is guaranteed by Assumption 1.1);
- let π_{s+1} be the concatenation of α_{s+1} , ω_{s+1} , β_{s+1} and θ_{s+1} .

We can check that π_{s+1} is indeed an SAW, and that $\pi_{s+1}(h_p^{s+1})$ is a closest vertex on π_{s+1} to $\phi_{s+1} \dots \phi_1 S_p$, for $s+2 \leq p \leq \delta n$, where $\pi_{s+1}(h_p^{s+1}) = \phi_{s+1}(\pi_s(h_p^s))$.

We repeat the above induction process until we construct the SAW $\pi_{\delta n}$. Note that $\pi_{\delta n}$ is an SAW whose length is at least n and at most $n + \delta n(N + 2N_0)$.

We consider the number of pairs (π, H) , and obtain

$$\boxed{\text{p1}} \quad (2.9) \quad |(\pi, H)| \geq \bar{c}_n(v) \binom{\kappa n}{\delta n}.$$

Moreover,

$$\boxed{\text{p2}} \quad (2.10) \quad |(\pi, H)| \leq \left(\sum_{t=n}^{n+\delta n(N+2N_0)} c_t(v) \right) [d^{|S|}|S|(2N_0 + N)^4]^{2\delta n}$$

where d is the maximal vertex degree of G ; d is obviously finite since G is quasi-transitive. Taking n th root of (2.9), (4), and letting $n \rightarrow \infty$, we have

$$(2.11) \quad \lambda(\tilde{E}_1) \frac{\kappa^\kappa}{\delta^\delta (\kappa - \delta)^{\kappa - \delta}} \leq \mu (d^{|S|}|S|(2N_0 + N)^4)^{2\delta}$$

but this is impossible when $\lambda(\tilde{E}_1) = \mu$, and $\frac{\delta}{\kappa} > 0$ and is sufficiently small. Therefore we have $\lambda(\tilde{E}_1) < \mu$. \square

To prove Theorem 1.3, we will analyze both the case $\lambda(E^*) = \mu$ and the case $\lambda(E^*) < \mu$.

We first assume that

$$\boxed{\text{1s}} \quad (2.12) \quad \lambda(E^*) = \mu.$$

Let $1 \leq k \leq |S|$. We make the following observations. First, $c_n(0, \tilde{E}_k)$ is a non-decreasing function of k ; secondly, if E^* does not occur on a given walk, then $E_{|S|}$ cannot occur. Therefore

$$c_n(0, E^*) \leq c_n(0, \tilde{E}_{|S|}) \leq c_n.$$

As a result, (2.12) implies

$$\limsup_{n \rightarrow \infty} c_n(0, \tilde{E}_{|S|})^{\frac{1}{n}} = \mu.$$

By Lemma 2.4, $\lambda(\tilde{E}_1) < \mu$. We may choose k with $1 \leq k < |S|$, such that

$$\boxed{\text{dk}} \quad (2.13) \quad \lambda(\tilde{E}_k) < \mu, \quad \lambda(\tilde{E}_{k+1}) = \mu$$

Let ϵ, m satisfy (2.1) and let $m \geq M$ satisfy (2.3). By Lemma 2.1, there exists $a = a(\epsilon, m) > 0$, such that

$$\boxed{\text{pst}} \quad (2.14) \quad \limsup_{n \rightarrow \infty} c_n(an, \tilde{E}_k(m))^{\frac{1}{n}} < \mu.$$

By quasi-transitivity of G , $\lambda(\tilde{E}_{k+1}) = \mu$ implies that there exists a vertex v_0 such that

$$\boxed{\text{ct1}} \quad (2.15) \quad \limsup_{n \rightarrow \infty} c_n^v(0, \tilde{E}_{k+1})^{\frac{1}{n}} = \mu.$$

Moreover, $\lambda(\tilde{E}_k) < \mu$ implies that for any vertex v of G ,

$$\boxed{\text{ct2}} \quad (2.16) \quad \limsup_{n \rightarrow \infty} c_n^v(an, \tilde{E}_k(m))^{\frac{1}{n}} < \mu,$$

where a, m satisfy (2.14).

Now assume that $\lambda(E^*) < \mu$, by quasi-transitivity of G , (1.2) and Lemma 2.1, for any vertex v of G ,

$$\boxed{\text{aml}} \quad (2.17) \quad \limsup_{n \rightarrow \infty} c_n^v(an, E^*(m))^{\frac{1}{n}} < \lim_{n \rightarrow \infty} c_n(v)^{\frac{1}{n}} = \mu.$$

Let π be an n -step SAW on G . We say that F occurs at the j th step of π if

- (a) if $\lambda(E^*) < \mu$, then
 - $E^*(m)$ occurs at the j th step of π ; and
 - assume that $\pi(j) \in \gamma S$, $\gamma \in \Gamma$ and the subwalk $(\pi(j-m), \dots, \pi(j+m))$ visits all the vertices of γS . Let $\pi(\alpha)$ ($\alpha \geq j-m$) be the first vertex of γS visited by π , and let $\pi(\beta)$ ($\alpha < \beta \leq j+m$) be the last vertex of γS visited by π . Then $\pi(\alpha-1)$ and $\pi(\beta+1)$ are in distinct components of $G \setminus \gamma S$.
- (b) if $\lambda(E^*) = \mu$, let k be as in (2.13), then
 - $\tilde{E}_k(m)$ occurs at the j th step, and \tilde{E}_{k+1} does not occur at the j th step; and
 - assume that $\pi(j) \in \gamma S$, $\gamma \in \Gamma$ and π visits exactly k vertices of γS . Let $\pi(\alpha)$ ($\alpha \geq j-m$) be the first vertex of γS visited by π , and let $\pi(\beta)$ ($\alpha < \beta \leq j+m$) be the last vertex of γS visited by π . Then $\pi(\alpha-1)$ and $\pi(\beta+1)$ are in distinct components of $G \setminus \gamma S$.

For $r \geq 0$, and $v \in V$

- if $\lambda(E^*) < \mu$, let $b_n^v(r, F)$ be the number of n -step SAWs on G starting from v , such that $E^*(m)$ occurs at least an times, and F occurs no more than r steps;

- if $\lambda(E^*) = \mu$, let $b_n^v(r, F)$ be the number of n -step SAWs on G starting from v , such that $\tilde{E}_k(m)$ occurs at least an times, E_{k+1} never occurs, and F occurs in no more than r steps.

125 **Lemma 2.5.** (a) If $\lambda(E^*) < \mu$, then for any $v \in V$, we have

$$\limsup_{n \rightarrow \infty} b_n^v(0, F)^{\frac{1}{n}} < \mu.$$

(b) If $\lambda(E^*) = \mu$, let v be a vertex satisfying (2.15), then

$$\limsup_{n \rightarrow \infty} b_n^v(0, F)^{\frac{1}{n}} < \mu.$$

Proof. Let v be a vertex satisfying the assumptions of the lemma. Assume that

$$\limsup_{n \rightarrow \infty} b_n^v(0, F)^{\frac{1}{n}} = \mu;$$

we will obtain a contradiction.

If $\lambda(E^*) < \mu$, let U_n be the set consisting of all the n -step SAWs on G starting from v such that $E^*(m)$ occurs at least an times, and F never occurs. .

If $\lambda(E^*) = \mu$, let k be given as in (2.13) and let U_n be the set consisting of all the n -step SAWs on G starting from v , such that $\tilde{E}_k(m)$ occurs at least an times, and \tilde{E}_{k+1} never occurs, and F never occurs.

Let $\pi \in U_n$.

Let j_1, \dots, j_u be the indices of the SAW π where $E^*(m)$ occurs if $\lambda(E^*) < \mu$ (where $E_k(m)$ occurs if $\lambda(E^*) = \mu$), such that

dil (2.18) $\text{dist}_G(\pi(j_i), \pi(j_l)) \geq 4(2m+1)|S|.$

We may assume $u = \kappa n$ such that (2.18) occurs. We choose a subset

$$H = \{t_1, \dots, t_{\delta n}\} \subset \{j_1, \dots, j_{\kappa n}\},$$

where $0 < \delta < \kappa$, $t_i < t_j$ when $i < j$, to perform the following inductive manipulations.

For $1 \leq i \leq \delta n$, let $\gamma_i S$ be the copy of S (where $\gamma_i \in \Gamma$) such that $\pi(t_i) \in \gamma_i S$, and all the vertices of $\gamma_i S$ are visited by π and by the subwalk $(\pi(t_i - m), \dots, \pi(t_i + m))$ if $\lambda(E^*) < \mu$ (exactly k vertices of $\gamma_i S$ are visited by the subwalk $(\pi(t_i - m), \dots, \pi(t_i + m))$ if $\lambda(E^*) = \mu$). Let $\pi(\alpha_i)$ ($\alpha_i \geq t_i - m$) be the first vertex of $\gamma_i S$ visited by π , and let $\pi(\beta_i)$ ($\alpha_i < \beta_i \leq t_i + m$) be the last vertex of $\gamma_i S$ visited by π .

Let A_i be the component of $G \setminus \gamma_i S$ containing $\pi(\alpha_i - 1)$ and $\pi(\beta_i + 1)$, and let $B_i, \phi(\gamma_i S, A_i)$ be as described in Assumption 1.1 (3). We construct a new SAW π_1 from π as follows.

- The subwalk $(\pi_1(0), \dots, \pi_1(\alpha_1))$ is the same as the subwalk $(\pi(0), \dots, \pi(\alpha_1))$.

- We map $(\pi(\beta_1 + 1), \dots, \pi(n))$, as an SAW in A_1 , to an SAW θ_1 in B_1 by $\phi(\gamma_1 S, A_1)$, as described in Assumption 1.1 (3).
- we use an SAW ω_1 in $G \setminus [A_1 \cup \phi(\gamma_1 S, A_1)]$ joining $\pi(\alpha_1)$ and the $\phi(\gamma_1 S)\pi(\beta_1)$. This is possible since S is connected by Assumption 1.1 (1), and $\gamma_1 S$ is an identical copy of S .
- Let π_1 be the concatenation of $(\pi_1(0), \dots, \pi_1(\alpha_1))$, ω_1 and θ_1 .

Let

$$\pi_0 := \pi, \quad \alpha_i^0 = \alpha_i, \quad \beta_i^0 = \beta_i.$$

Assume that we have constructed SAWs π_1, \dots, π_k , where $1 \leq k < \delta n$. Assume that $\eta_k = (\pi_{k-1}(0), \dots, \pi_{k-1}(\alpha_k^{k-1}))$ is a subwalk of both π_{k-1} and π_k , and that $\pi_{k-1} \setminus \eta_k$ is mapped by a graph automorphism $\phi_k \in \Gamma$ to a subwalk of π_k . Let

$$\begin{aligned} \alpha_i^k &= \alpha_i^{k-1}, & \text{for } 1 \leq i \leq k; \\ \beta_j^k &= \beta_j^{k-1}, & \text{for } 1 \leq j \leq k-1. \end{aligned}$$

Let α_i^k ($k+1 \leq i \leq \delta n$) (resp. β_j^k ($k \leq j \leq \delta n$)) be the step in π_k of the image of $\pi_{k-1}(\alpha_i^{k-1})$ (resp. $\pi_{k-1}(\beta_j^{k-1})$) under the graph isomorphism ϕ_k .

Now we construct an SAW π_{k+1} , following the procedure below.

- Let η_{k+1} be the subwalk of π_k from $\pi_k(0)$ to $\pi_k(\alpha_{k+1}^k)$.
- Let $\tilde{S}_{k+1} := \phi_k \dots \phi_1 S_{k+1}$. Let A_{k+1} be the component of $G \setminus \tilde{S}_{k+1}$ containing $\pi_k(\alpha_{k+1}^k)$; use $\phi_{k+1} := \phi(\tilde{S}_{k+1}, A_{k+1})$ to map the subwalk $\pi_k(\beta_{k+1}^k), \dots, \pi_k(n - \beta_{k+1}^k + \beta_{k+1}^k)$ to an SAW θ_{k+1} in another component of $G \setminus \tilde{S}_{k+1}$.
- Use an SAW in $G \setminus [A_{k+1} \cup \phi_{k+1}(\tilde{S}_{k+1}, A_{k+1})A_{k+1}]$ to join $\pi_k(\alpha_{k+1}^k)$ and $\phi_{k+1}(\pi_k(\beta_{k+1}^k))$ (note that the existence of such an SAW is guaranteed by Assumption 1.1);
- let π_{k+1} be the concatenation of η_{k+1} , ω_{k+1} and θ_{k+1} .

We can check that π_{k+1} is indeed an SAW.

We claim that

$$\boxed{\text{ie}} \quad (2.19) \quad [\phi_k \dots \phi_1 S_{k+1}] \cap [\phi_{k+1} \dots \phi_1 S_{k+2}] = \emptyset.$$

To see why (2.19) is true, note that by (2.18), we have

$$\text{dist}(S_{k+1}, S_{k+2}) \geq 2|S|.$$

Since ϕ_i ($1 \leq i \leq k$)'s are automorphisms of G , we have

$$\text{dist}(\phi_k \dots \phi_1 S_{k+1}, \phi_k \dots \phi_1 S_{k+2}) \geq 2|S|.$$

This implies

$$[\phi_k \dots \phi_1 S_{k+1}] \cap [\phi_k \dots \phi_1 S_{k+2}] = \emptyset.$$

Since ϕ_{k+1} maps $\phi_k \dots \phi_1 S_{k+2}$ from one component of $G \setminus [\phi_k \dots \phi_1 S_{k+1}]$ to another component of $G \setminus [\phi_k \dots \phi_1 S_{k+1}]$, (2.19) follows.

We continue the above construction process until we have already constructed the SAW $\pi_{\delta n}$. Counting the number of pairs $|(\pi, H)|$, we have

$$|(\pi, H)| \geq |U_n| \binom{\kappa n}{\delta n},$$

and

$$|(\pi, H)| \leq \left(\sum_{i=1}^{n+\delta n N} c_i \right) \left(\sum_{i=1}^{2m} c_i d^{|S|} |S| (N + d^{|S|})^3 \right)^{\delta n}.$$

We have

$$\limsup_{n \rightarrow \infty} |U_n|^{\frac{1}{n}} \leq \frac{\delta^\delta (\kappa - \delta)^{\kappa - \delta}}{\kappa^\kappa} \mu^{1+2N\delta} \left[\sum_{i=1}^{2m} c_i d^{|S|} |S| (N + d^{|S|})^3 \right]^\delta := g(\delta)$$

Note that

$$g(0) = \mu, \quad g'(0) = -\infty.$$

Hence for $0 < \delta \leq \delta_0$, $g(\delta) < \mu$. Therefore we have

$$\limsup_{n \rightarrow \infty} |U_n|^{\frac{1}{n}} < \mu,$$

and the proof is complete. □

Lemma 2.6. For $r \geq 0$, let

$$b_n(r, F) = \sup_{v \in G} b_n^v(r, F).$$

Then

$$\boxed{\text{lm}} \quad (2.20) \quad \limsup_{n \rightarrow \infty} b_n(0, F)^{\frac{1}{n}} < \mu.$$

Proof. If $\lambda(E^*) < \mu$, then (2.20) follows from Part (a) of Lemma 2.5, and the quasi-transitivity of G under the action of Γ .

Now assume that $\lambda(E^*) = \mu$. Let k be a positive integer satisfying (2.13). We consider the following cases

A. The starting vertex v of SAWs satisfy

$$\limsup_{n \rightarrow \infty} c_n^v(0, \tilde{E}_{k+1})^{\frac{1}{n}} = \mu.$$

B. The starting vertex v of SAWs satisfy

$$\boxed{\text{clm}} \quad (2.21) \quad \limsup_{n \rightarrow \infty} c_n^v(0, \tilde{E}_{k+1})^{\frac{1}{n}} < \mu.$$

In Case A., (2.20) follows from Part (b) of Lemma 2.5. In Case B., (2.20) follows from (2.21) and the following fact

$$b_n^v(0, F) \leq c_n^v(0, \tilde{E}_{k+1}).$$

□

1f **Lemma 2.7.** *There exist $a', R' \in (0, 1)$, such that*

$$\limsup_{n \rightarrow \infty} b_n(a'n, F)^{\frac{1}{n}} < R'\mu.$$

Proof. Let us consider the number of n -step SAWs on G starting from a fixed vertex, such that F occurs less than ξn times, where ξ is a small positive number.

Let ϵ, m satisfy (2.3). If F occurs in less than ξn times, then it occurs in less than ξn of the $N = \lfloor \frac{n}{m} \rfloor$ m -step subwalks

$$(\pi(im + 1), \dots, \pi((i + 1)m)), \quad \text{for } 0 \leq i \leq n.$$

Hence we have

$$b_n(\xi n, F) \leq \sum_{i=1}^{\xi n} \binom{N}{i} [\mu(1 - \epsilon)]^{Nm - im} (\mu(1 + \epsilon))^{im} c_{n - Nm}.$$

Note that when $\lambda(E^*) < \mu$, $[\mu(1 - \epsilon)]^m$ gives an upper bound for the number of m -step SAWs starting from a fixed vertex, such that one of the followings holds

- $E^*(m)$ occurs less than am times; or
- $E^*(m)$ occurs at least am times, and F never occurs,

where a, m satisfies (2.17).

When $\lambda(E^*) = \mu$, $[\mu(1 - \epsilon)]^m$ gives an upper bound for the number of m -step SAWs starting from a fixed vertex such that

- $\tilde{E}_k(m)$ occurs less than am times; or
- $\tilde{E}_k(m)$ occurs at least am times, E_{k+1} never occurs, and F never occurs.

Here k satisfies (2.13), and a, m satisfies (2.14).

When ξ is sufficiently small, we have

$$\limsup_{n \rightarrow \infty} b_n(\xi n, F)^{\frac{1}{n}} < \mu,$$

and the lemma follows. □

Proof of Theorem 1.3 A. If $\lambda(E^*) < \mu$, let d_n^v be the number of n -step SAWs in G starting from v such that $E^*(m)$ occurs at least an times, and F occurs at least $a'n$ times, where a is given by (2.17) and a' is given by Lemma 2.7. Then we have

pt1 (2.22)
$$\limsup_{n \rightarrow \infty} d_n^v = \mu.$$

Now assume that $\lambda(E^*) = \mu$. Let v be a vertex of G satisfying (2.15). Let $a > 0$ be given by (2.16), and $a' > 0$ be given by Lemma 2.7. Let d_n^v be the number of n -step SAWs in G such that E_{k+1} never occurs, $E_k(m)$ occurs at least an times, and F occurs at least $a'n$ times. Then

$$\boxed{\text{pt2}} \quad (2.23) \quad \limsup_{n \rightarrow \infty} d_n^v = \mu.$$

Note that in an SAW π , if F occurs at least $a'n$ times, then $\|\pi\| \geq a'n$. The theorem follows from (2.22) and (2.23). \square

3. PROOF OF THEOREM 1.3 B.

$\boxed{\text{ppb}}$

We prove Theorem 1.3 B. in this section. Let $G = (V, E)$ be a graph satisfying Assumption 1.2. Since any graph satisfying Assumption 1.2 must satisfy Assumption 1.1 as well, all the results proved in Section 2 also apply to graphs satisfying Assumption 1.2.

Let π be an n -step SAW on G . Recall that E^* occurs at the j th step of π if there exists $\gamma S \in \Gamma S$ such that $\pi(j) \in \gamma S$, and all the vertices of γS are visited by π . For $k \geq 1$, we say that \mathcal{E}_k occurs at the j th step of π , if there exists $\gamma S' \in \Gamma S'$, such that $\pi(j) \in \gamma S'$, and at least k vertices of $\gamma S'$ are visited by π , where S' containing S is given as in Assumption 1.2. We say that $\tilde{\mathcal{E}}_k$ occurs at the j th step of π if E^* or \mathcal{E}_k (or both) occur there.

In the following, we will use \mathcal{E} to denote any of E^* , \mathcal{E}_k or $\tilde{\mathcal{E}}_k$. If m is a positive integer, we say that $\mathcal{E}(m)$ occurs at the j th step of π if \mathcal{E} occurs at the m th step of the $2m$ -step subwalk $(\pi(j-m), \dots, \pi(j+m))$.

For $r \geq 0$ and $v \in G$, let $c_n^v(r, \mathcal{E})$ (resp. $c_n^v(r, \mathcal{E}(m))$) be the number of n -step SAWs starting from v for which \mathcal{E} (resp. $\mathcal{E}(m)$) occurs at no more than r different steps. Let

$$\begin{aligned} c_n(r, \mathcal{E}) &= \sup_{v \in V} c_n^v(r, \mathcal{E}) \\ c_n(r, \mathcal{E}(m)) &= \sup_{v \in V} c_n^v(r, \mathcal{E}(m)) \end{aligned}$$

Let

$$\lambda(\mathcal{E}) = \limsup_{n \rightarrow \infty} c_n(0, \mathcal{E})^{\frac{1}{n}}.$$

Let μ be the connective constant of G . We have that $\lambda(\mathcal{E}) < \mu$ if and only if

$$\boxed{\text{eem1}} \quad (3.1) \text{ there exist } \epsilon > 0, M \in \mathbb{N}, \text{ such that } c_m(0, \mathcal{E}) < [\mu(1 - \epsilon)]^m, \text{ for } m \geq M.$$

$\boxed{111}$ **Lemma 3.1.** *Let k satisfy $1 \leq k \leq |S|$, and*

$$\boxed{11k} \quad (3.2) \quad \lambda(\mathcal{E}) < \mu.$$

Let ϵ, M satisfy (3.1), and let $m \geq M$ satisfy

$$\boxed{\text{eem2}} \quad (3.3) \quad c_m \leq [\mu(1 + \epsilon)]^m.$$

Then there exist $a = a(\epsilon, m)$ and $R = R(\epsilon, m) \in (0, 1)$, such that

$$\limsup_{n \rightarrow \infty} c_n(an, \mathcal{E}(m))^{\frac{1}{n}} < R\mu.$$

Proof. Same as the proof of Lemma 2.1. □

$$\boxed{112} \quad \textbf{Lemma 3.2.} \quad \lambda(\tilde{\mathcal{E}}_1) < \mu.$$

Proof. The lemma follows from Lemma 2.4 and the fact that $\lambda(\tilde{\mathcal{E}}_1) < \lambda(\tilde{E}_1)$. □

$$\boxed{133} \quad \textbf{Lemma 3.3.} \quad \lambda(E^*) < \mu.$$

Proof. Assume that

$$\boxed{134} \quad (3.4) \quad \lambda(E^*) = \mu;$$

we will obtain a contradiction.

We make the following observations. First, $c_n(0, \tilde{\mathcal{E}}_k)$ is a non-decreasing function of k ; secondly, if E^* does not occur on a given walk, then $\mathcal{E}_{|S'|}$ cannot occur. Therefore

$$c_n(0, \mathcal{E}^*) \leq c_n(0, \tilde{\mathcal{E}}_{|S'|}) \leq c_n.$$

As a result, (3.4) implies

$$\limsup_{n \rightarrow \infty} c_n(0, \tilde{\mathcal{E}}_{|S'|})^{\frac{1}{n}} = \mu.$$

By Lemma 3.2, $\lambda(\tilde{\mathcal{E}}_1) < \mu$. We may choose k with $1 \leq k < |S|$, such that

$$\boxed{\text{me1}} \quad (3.5) \quad \lambda(\tilde{\mathcal{E}}_k) < \mu, \quad \lambda(\tilde{\mathcal{E}}_{k+1}) = \mu.$$

Let ϵ, m satisfy (3.1) and let $m \geq M$ satisfy (3.3). By Lemma 3.1, there exists $a = a(\epsilon, m) > 0$, such that

$$\boxed{\text{me2}} \quad (3.6) \quad \limsup_{n \rightarrow \infty} c_n(an, \tilde{\mathcal{E}}_k(m))^{\frac{1}{n}} < \mu.$$

By (3.5) and (3.6) and the quasi-transitivity of G , there exists a vertex v_0 of G , such that

$$\boxed{35} \quad (3.7) \quad \limsup_{n \rightarrow \infty} c_n^{v_0}(0, \tilde{\mathcal{E}}_{k+1})^{\frac{1}{n}} = \mu;$$

and for any $v \in V$,

$$\boxed{36} \quad (3.8) \quad \limsup_{n \rightarrow \infty} c_n^v(an, \tilde{\mathcal{E}}_k(m))^{\frac{1}{n}} < \mu.$$

Let T_n be the set consisting of all the n -step SAWs starting from v_0 for which \mathcal{E}_{k+1} never occurs, but $\tilde{\mathcal{E}}_k(m)$ occurs at least an times. We have that

$$|T_n| \geq c_n^{v_0}(0, \tilde{\mathcal{E}}_{k+1}) - c_n^{v_0}(an, \tilde{E}_k(m)).$$

By (3.7) and (3.8), we have

$$\boxed{\text{tn}} \quad (3.9) \quad \limsup_{n \rightarrow \infty} |T_n|^{\frac{1}{n}} = \mu.$$

The rest of the proof is devoted to show the existence of $t < 1$, such that

$$\limsup_{n \rightarrow \infty} |T_n|^{\frac{1}{n}} < t\mu.$$

This contradicts (3.9), and the lemma follows.

Let d be the maximal vertex degree of G . Let $\pi \in T_n$, so that π contains at least an occurrences of $\tilde{\mathcal{E}}_k(m)$. We can find $j_1 < \dots < j_u$ with $u = \lfloor \kappa n \rfloor - 2$, where

$$\boxed{\text{ka}} \quad (3.10) \quad \kappa = \frac{a}{(2m+2)d^{4|S'|}},$$

such that

$\tilde{\mathcal{E}}_k(m)$ occurs at the $j_1\text{th}, j_2\text{th}, \dots, j_u\text{th}$ steps of π ,

(and perhaps other steps as well), and in addition,

- $0 < j_1 - m, j_u + m < n$,
- $j_t + m < j_{t+1} - m$;
- for any $\gamma_1, \gamma_2 \in \Gamma$, such that $\pi(j_i) \in \gamma_1 S'$ and $\pi(j_l) \in \gamma_2 S'$, $i \neq l$, $\gamma_1 S'$ and $\gamma_2 S'$ are disjoint.

Such j_t 's may be found by the following iterative construction. First j_1 is the smallest $j > m$ such that $\mathcal{E}_k(m)$ occurs at the j th step of π . Having found j_1, j_2, \dots, j_r , let j_{r+1} be the smallest j such that

- (a) $j_r + m < j_{r+1} - m$;
- (b) $\tilde{E}_k(m)$ occurs at the j th step of π ;
- (c) for any $\gamma_1, \gamma_2 \in \Gamma$, such that $\pi(j_i) \in \gamma_1 S'$ ($i \leq r$), and $\pi(j_{r+1}) \in \gamma_2 S'$, we have $\gamma_1 S' \cap \gamma_2 S' = \emptyset$.

Condition (a) gives rise to the factor $(2m+2)$ in the denominator of (3.10), and Condition (c) gives rise to the factor $d^{4|S'|}$.

Let $t \in \{1, 2, \dots, u\}$. Since $\tilde{\mathcal{E}}_k(m)$ but not $\tilde{\mathcal{E}}_{k+1}$ occurs at the j_t th step, π visits at most k vertices in each γS containing $\pi(j_t)$. Let Ψ_t be the set consisting of all the copies $\gamma S'$ of S' in $\Gamma S'$, such that $\pi(j_t) \in \gamma S'$, π visits exactly k vertices of $\gamma S'$, and these k vertices lie between $\pi(j_t - m)$ and $\pi(j_t + m)$ on π . Choose $\gamma_t S \in \Psi_t$. For $t = 1, 2, \dots, u$, let

$$\alpha_t = \min\{i : \pi(i) \in \gamma_t S'\}, \quad \beta_t = \max\{i : \pi(i) \in \gamma_t S'\},$$

so that

$$j_t - m \leq \alpha_t \leq j_t \leq \beta_t \leq j_t + m.$$

We next describe the strategy for the replacement of the subwalk $(\pi(\alpha_t), \pi(\alpha_t + 1), \dots, \pi(\beta_t))$. Starting from $\pi(\alpha_t)$, the walk follows an SAW in $\gamma_t S$ joining $\pi(\alpha_t)$ and $\pi(\beta_t)$, and visits every vertex in $\gamma_t S$. Such an SAW exists by Assumption 1.2.

Let δ satisfy $0 < \delta < \kappa$, to be chosen later, and set $s = \delta n$. Let $H = (h_1, \dots, h_s)$ be an oriented subset of $\{j_1, \dots, j_u\}$. We shall make an appropriate substitution in the neighborhood of each $\pi(h_t)$ to obtain an SAW $\pi^* = (\pi, H)$.

We estimate the number of pairs (π, H) as follows. First, the number $|(\pi, H)|$ is at least the cardinality of T_n multiplied by the minimum number of possible choices of H as π ranges over T_n . Any subset of $\{j_1, \dots, j_u\}$ with cardinality $s = \delta n$ may be chosen for H , whence

$$\boxed{\text{lb}} \quad (3.11) \quad |(\pi, H)| \geq |T_n| \binom{\kappa n - 2}{\delta n}$$

We bound (π, H) above by counting the number of SAWs π_* of G with length not exceeding $n + |S'| \delta n$, and multiplying by an upper bound for the number of pairs (π, H) giving rise to a particular π_* .

The number of possible choices of π_* is no greater than $\sum_{i=0}^{n+|S'| \delta n} c_i$. A given π_* contains $|H| = \delta n$ occurrences of visits to all the vertices of some copy of S' under Γ . At the t th such occurrence, $\pi(h_t)$ is a point on $\gamma_t S'$ and there are no more than $|S'|$ different choices of $\pi(h_t)$. For given π_* and $(\pi(h_t) : t = 1, 2, \dots, s)$, there are at most $[\sum_{i=1}^{2m} c_i]^{\delta n}$ corresponding SAWs π of G . Therefore

$$\boxed{\text{ub}} \quad (3.12) \quad |(\pi, H)| \leq \left(d^{|S'|} |S'|^2 \sum_{i=1}^{2m} c_i \right)^{\delta n} \left(\sum_{i=0}^{n+|S'| \delta n} c_i \right)$$

We combine (3.11) and (3.12), take n th root and the limit as $n \rightarrow \infty$, we obtain, by the fact that $c_n^{\frac{1}{n}} \rightarrow \mu$,

$$\mu \frac{\kappa^\kappa}{\delta^\delta (\kappa - \delta)^{\kappa - \delta}} \leq \left(d^{|S'|} |S'|^2 \sum_{i=1}^{2m} c_i \right)^\delta \mu^{1+|S'| \delta}.$$

There exists $Z = Z(\epsilon, m, d, |S'|)$, such that

$$d^{|S'|} |S'|^2 \mu^{|S'|} \sum_{i=1}^{2m} c_i \leq Z,$$

therefore

$$\mu \leq f(\eta)^\kappa \mu,$$

where $f(\eta) = Z^\eta \eta^\eta (1 - \eta)^{1-\eta}$, and $\eta = \frac{\delta}{\kappa}$. Since

$$\lim_{\eta \rightarrow 0} f(\eta) = 1, \quad \lim_{\eta \rightarrow 0} f'(\eta) = -\infty,$$

we have $f(\eta) < 1$ for sufficiently small $\eta = \eta(Z) > 0$. The contradiction implies the lemma. \square

Proof of Theorem 1.3 B. By Lemma 3.3, $\lambda(E^*) < \mu$. Let σ_n be the number of n -step SAWs on G starting from a fixed vertex such that one of the followings holds

- $E^*(m)$ occurs less than an times; or
- $E^*(m)$ occurs at least an times, and F occurs less than $a'n$ times

where a satisfies (2.17) and a' satisfies Lemma 2.7. We have

$$\limsup_{n \rightarrow \infty} \sigma_n < \mu.$$

For any n -step SAW π on G starting from a fixed vertex not counted in σ_n , F occurs at least $a'n$ times, and therefore $\|\pi\| \geq a'n$. This implies that SAWs on G have positive speed. \square

4. GROUPS WITH MORE THAN ONE END

p2

In this section, we prove Theorem 1.7. The proof is based on the stalling's splitting theorem, and an explicit construction of the set S satisfying Assumption 1.1.

141

Lemma 4.1. *Let $G = (V, E)$ be an infinite, connected, locally finite graph with maximal vertex degree d . Let $A, B \subseteq V$ be two finite set of vertices of G satisfying $A \subseteq B$. Let $G \setminus A$ (resp. $G \setminus B$) be the subgraph of G obtained from G by removing all the vertices in A (resp. B) as well as their incident edges. If $G \setminus A$ has at least two infinite components, then $G \setminus B$ has at least two infinite components.*

Proof. Let R_1 and R_2 be two infinite components of $G \setminus A$. We will show that each one of $R_1 \setminus B$ and $R_2 \setminus B$ has at least one infinite component.

Since $R_1 \setminus B$ can be obtained from R_1 by removing finitely many vertices and edges, if $R_1 \setminus B$ has no infinite components, then $R_1 \setminus B$ has infinitely many finite components. Moreover, since each vertex of G is incident to at most d edges, for any connected subgraph of G , removing one vertex of the subgraph as well as all its incident edges can split the subgraph into at most d connected components. Since $|B| < \infty$, it is not possible that $R_1 \setminus B$ has infinitely many finite components. As a result, $R_1 \setminus B$ has at least one infinite component. The fact that $R_2 \setminus B$ has at least one infinite component can be proved similarly. \square

Now we prove Theorem 1.7. By Stalling's splitting theorem ([26, 27]), a group Γ has more than one end if and only if one of the followings holds.

- (1) the group Γ is an amalgated free product, i.e.

$$\Gamma = H *_C K,$$

where H, K are groups, and C is a finite group such that $C \neq H$ and $C \neq K$.

- (2) There exists a group H , two finite subgroups C_1, C_2 of H , and a group isomorphism $\phi : C_1 \rightarrow C_2$, such that Γ is the following HNN extension

$$\Gamma = \langle H, t | t^{-1}c_1t = \phi(c_1), \forall c_1 \in C_1 \rangle$$

We will prove Theorem 1.7 for Case (1) and (2), in the subsections below.

pf141

4.1. Proof of Theorem 1.7 when Γ is an amalgated free product. In this section, we prove Theorem 1.7 when the group Γ is a free product with amalgamation, as described in Case (1). We start with the following standard result concerning members in a amalgamated free product.

123

Lemma 4.2. *(Normal form for amalgamated free product [20]) Every element in $H *_C K$ which is not in the image of C can be written in the normal form*

$$v_1 \cdot \dots \cdot v_n$$

where the terms v_i lie in $H \setminus C$ or $K \setminus C$ and alternate between two sets. The length n is uniquely determined and two such expressions $v_1 \cdot \dots \cdot v_n$ give the same element in $H *_C K$ if and only if there are elements $c_1, \dots, c_n \in C$, so that

$$w_k = c_{k-1}v_kc_k^{-1},$$

where $c_0 = c_n = 1$.

Assume that Γ is a free product with amalgamation as described in Case (1). Let G_H (resp. G_K) be a locally finite Cayley graph for H (resp. K) with respect to a finite set of generators T_H (resp. T_K). Let G_0 be a locally finite Cayley graph of Γ constructed from G_H and G_K as follows.

- (a) Construct the free product graph $G_H * G_K$ of G_H and G_K . In other words $G_H * G_K$ is the Cayley graph for the free product $H * K$ with respect to the generator set $T_H \cup T_K$.
- (b) Glue vertices $u \in G_H * G_K$ and $w \in G_H * G_K$ if there exists a vertex $v \in G_H * G_K$ such that $v^{-1}u = v^{-1}w \in C$.

Let 1_Γ be the identity element of the group Γ . Let G be a locally finite Cayley graph of Γ with respect to the generator set T such that $|T| < \infty$, $T = T^{-1}$ and $1_\Gamma \notin T$. Let

dd

$$(4.1) \quad D_0 = \max_{t \in T, v \in \Gamma} \text{dist}_{G_0}(v, vt),$$

where $\text{dist}_{G_0}(\cdot, \cdot)$ is the graph distance in G_0 .

Let

$$\boxed{\text{dc}} \quad (4.2) \quad C_1 = \{v \in \Gamma, \text{dist}_{G_0}(v, C) \leq D_0 + 1\}.$$

Then $C \subseteq C_1$, and $|C_1| < \infty$. Let C_2 be a finite set of vertices containing C_1 , such that C_2 is connected in G in the following sense for any $u, v \in C_2$ there exists a path

$$w_0(=u), w_1, \dots, w_{n-1}, w_n(=v),$$

such that for $0 \leq i \leq n$, $w_i \in C_2$; for $1 \leq i \leq n$, w_{i-1} and w_i are adjacent vertices in G .

$\boxed{\text{luc}}$ **Lemma 4.3.** *Let $u \in \Gamma$ have normal form starting from a term in $H \setminus C$; and let $v \in \Gamma$ have normal form starting from a term in $K \setminus C$, then any path in G_0 joining 1_Γ and $u^{-1}v$ must visit a point in $u^{-1}C$.*

Proof. Since $u \in \Gamma$ has a normal form starting from a term in $H \setminus C$, and $v \in \Gamma$ has a normal form starting from a term in $K \setminus C$, then the concatenation of normal forms of u^{-1} and v gives us a normal form of $u^{-1}v$, denoted by $z_1 \cdot \dots \cdot z_n$. This normal form gives rise to a path $\tau_{u^{-1}v}$ in G_0 joining 1_Γ and $u^{-1}v$. More precisely, $\tau_{u^{-1}v}$ is the concatenation of n paths τ_i , $1 \leq i \leq n$, such that τ_i is the shortest path in $z_1 \cdot \dots \cdot z_{i-1}H$ (resp. $z_1 \cdot \dots \cdot z_{i-1}K$) joining $z_1 \cdot \dots \cdot z_{i-1}$ and $z_1 \cdot \dots \cdot z_i$, if $z_i \in H \setminus C$ (resp. $z_i \in K \setminus C$). We can see that the path $\tau_{u^{-1}v}$ visits the vertex u^{-1} .

Assume that $l_{u^{-1}v}$ is a path in G_0 joining 1_Γ and $u^{-1}v$. Then $l_{u^{-1}v}$ gives rise to a sequence x_1, \dots, x_m , such that

- (a) for $1 \leq i \leq m$, $l_{u^{-1}v}$ visits every vertex in $\{x_1 \cdot \dots \cdot x_i\}_{1 \leq i \leq m}$;
- (b) for $1 \leq i \leq m$, $x_i \in H \setminus C$ or $x_i \in K \setminus C$;
- (c) for $1 \leq i \leq m-1$, if $x_i \in H \setminus C$, then $x_{i+1} \in K \setminus C$;
- (d) $u^{-1}v = x_1 \cdot \dots \cdot x_m c$, where $c \in C$.

Indeed x_1, \dots, x_m can be found as follows. Let $\{y_1, \dots, y_l\}$ be all the vertices incident to an edge in $H \setminus C$ along $l_{u^{-1}v}$ and an edge in $K \setminus C$ along $l_{u^{-1}v}$, and assume that starting from 1_Γ and traversing along $l_{u^{-1}v}$, one visits y_1, \dots, y_l in order. Let $y_0 = 1_\Gamma$. From $\{y_1, \dots, y_l\}$, we perform the following manipulations.

- A. Remove all the y_i 's such that $y_{i-1}^{-1}y_i \in C$; let $\{y_1^{(1)}, \dots, y_{l_1}^{(1)}\}$ be the remaining set of vertices;
- B. Remove all the $y_i^{(1)}$'s such that either both $[y_{i-1}^{(1)}]^{-1}y_i^{(1)}$ and $[y_i^{(1)}]^{-1}y_{i+1}^{(1)}$ are in H or both $[y_{i-1}^{(1)}]^{-1}y_i^{(1)}$ and $[y_i^{(1)}]^{-1}y_{i+1}^{(1)}$ are in K ; let $\{y_1^{(2)}, \dots, y_{l_2}^{(2)}\}$ be the remaining set of vertices.

Once we have constructed $\{y_1^{(2j)}, \dots, y_{l_2}^{(2j)}\}$, for $j \geq 1$, we perform the following manipulations.

- A. Remove all the y_i 's such that $[y_{i-1}^{(2j)}]^{-1}y_i^{(2j)} \in C$; let $\{y_1^{(2j+1)}, \dots, y_{l_1}^{(2j+1)}\}$ be the remaining set of vertices;

- B. Remove all the $y_i^{(1)}$'s such that either both $[y_{i-1}^{(2j+1)}]^{-1}y_i^{(2j+1)}$ and $[y_i^{(2j+1)}]^{-1}y_{i+1}^{(2j+1)}$ are in H or both $[y_{i-1}^{(2j+1)}]^{-1}y_i^{(2j+1)}$ and $[y_i^{(2j+1)}]^{-1}y_{i+1}^{(2j+1)}$ are in K ; let $\{y_1^{(2j+2)}, \dots, y_{l_{2j+2}}^{(2j+2)}\}$ be the remaining set of vertices.

We repeat the process above until we end up with a set of vertices $\{y_1^{(2k)}, \dots, y_{l_{2k}}^{(2k)}\}$ satisfying

- (a) for $1 \leq i \leq l_{2k}$, $[y_{i-1}^{(2k)}]^{-1}y_i^{(2k)} \in H \setminus C$ or $[y_i^{(2k)}]^{-1}y_{i+1}^{(2k)} \in K \setminus C$;
- (b) for $1 \leq i \leq l_{2k} - 1$, if $[y_{i-1}^{(2k)}]^{-1}y_i^{(2k)} \in H \setminus C$, then $[y_i^{(2k)}]^{-1}y_{i+1}^{(2k)} \in K \setminus C$;
- (c) $u^{-1}v = y_{l_{2k}}^{(2k)}c$, where $c \in C$.

Obviously the process above will terminate in finitely many steps. Let $x_1 = y_1^{(2k)}$, for $2 \leq i \leq l_{2k}$, $x_i = [y_{i-1}^{(2k)}]^{-1}y_i^{(2k)}$. Then

$$x_1 \cdot x_2 \cdot \dots \cdot [x_{l_{2k}}c]$$

gives another normal form for $u^{-1}v$. By the uniqueness of normal form Lemma 4.2, we have $l_{2k} = m = n$, and

$$z_k = c_{k-1}^{-1}x_k c_k.$$

where $c_k \in C$. Since $l_{u^{-1}v}$ visits every vertex in $\{x_1 \cdot \dots \cdot x_i\}$, $1 \leq i \leq n$, $l_{u^{-1}v}$ visits a vertex in $u^{-1}C$. \square

143 **Lemma 4.4.** *Let $u \in \Gamma$ have normal form starting from a term in $H \setminus C$; and let $v \in \Gamma$ have normal form starting from a term in $K \setminus C$. If the lengths of normal forms of u and v are sufficiently large, u and v are in two distinct infinite components of $G \setminus C_1$.*

In particular, $G \setminus C_1$ has at least two distinct infinite components.

Proof. By Lemma 4.3, any path in G_0 joining 1_Γ and $u^{-1}v$ must visit a point in $u^{-1}C$.

Assume that there exists a path l_{uv} joining u and v in $G \setminus C_1$. Then for any edge $\langle p, q \rangle$ along l_{uv} , p and q can be joined by a path in G_0 which does not pass through any vertex in C . That is because $\text{dist}_{G_0}(p, q) \leq D_0$ by (4.1), and $\text{dist}_{G_0}(p, C) \geq D_0 + 2$ and $\text{dist}_{G_0}(q, C) \geq D_0 + 2$, by (4.2) and the fact that $p, q \in G \setminus C_1$. By replacing each edge along l_{uv} with a path in G_0 joining p and q that does not pass through any vertex in C , we obtain that the shortest path in G_0 joining u and v does not pass through any vertex in C . This implies that there exists a path in G_0 joining 1_Γ and $u^{-1}v$ which does not visit any vertex of $u^{-1}C$, by the vertex-transitivity of G_0 . This is a contradiction.

As a result, any path in G joining u and v must visit a vertex in C_1 . Since C_1 is finite, the lengths of normal forms for vertices in C_1 have a maximum M_0 . If the lengths of normal forms for u and v exceed M_0 , then $u \in G \setminus C_1$ and $v \in G \setminus C_1$. This

means that u and v are in two distinct components of $G \setminus C_1$. Moreover, we can find a singly-infinite path l_u (resp. l_v) on G starting from u (resp. v), such that moving along the path from u , the length of normal forms along the path is non-decreasing. Therefore all the vertices along l_u (resp. l_v) are in $G \setminus C_1$. As a result, u and v are in two distinct infinite component of $G \setminus C_1$. \square

Let Ω_n be the set of n -step SAWs on G starting from the identity vertex 1_Γ . Let $r_1 \in \Gamma$ have a normal form starting from a term in $H \setminus C$, and ending in a term in $H \setminus C$. Let $r_2 \in \Gamma$ have a normal form starting from a term in $K \setminus C$ and ending at a term in $K \setminus C$. Assume that the lengths of normal forms of r_1 and r_2 are strictly greater than the maximal length of normal forms of elements in C_2 . Also assume that

$$\boxed{\text{dr1}} \quad (4.3) \quad \text{dist}_G(r_1, C_2) \geq |C_2| + 1;$$

$$\boxed{\text{dr2}} \quad (4.4) \quad \text{dist}_G(r_2, C_2) \geq |C_2| + 1.$$

Let A (resp. B) be the collection of elements in Γ with a normal form starting from an element in $H \setminus C$ (resp. $K \setminus C$). Let

$$\boxed{\text{cs}} \quad (4.5) \quad S = C_2,$$

then by the construction of C_2 it is obvious that S is connected. The graph $G \setminus S$ has at least two distinct infinite components by Lemma 4.4, $C_1 \subset C_2$ and Lemma 4.1. Let $A_1 \subseteq A$ be a component of $G \setminus S$, define $\phi(S, A_1) = r_2$. Let $B_1 \subseteq B$ be a component of $G \setminus S$, define $\phi(S, B_1) = r_1$.

Lemma 4.5. *Assumption 1.1 (3) holds with the choice of S as given by (4.5).*

Proof. For each component of $G \setminus S$, either all the elements in the component have a normal form starting from a term in $H \setminus C$, or all the elements in the component have a normal form starting from a term in $K \setminus C$, by Lemma 4.4.

Let A_1 be a component of $G \setminus S$ such that all the elements in A_1 have a normal form starting from a term in $H \setminus C$. For each $v \in A_1$, since r_2 has a normal form ending in a term in $K \setminus C$, and v has a normal form starting from a term in $H \setminus C$, the concatenation of normal forms of r_2 and v gives us a normal form of $r_2 v$. Moreover $r_2 v$ has a normal form starting from a term in $K \setminus C$ since r_2 has a normal form starting from a term in $K \setminus C$. Therefore, $r_2 v \notin A_1$. Given the assumption that the length of the normal form of r_2 is strictly greater than the maximal length of normal forms of elements in S , the length of the normal form of $r_2 v$ is also strictly greater than the maximal length of normal forms of elements in S , we have $r_2 v \notin S$. Hence $r_2 v$ is in a component of $G \setminus S$ different from A_1 . For any two vertices $u, v \in A_1$, by the connectivity of A_1 , there exists a path l_{uv} joining u and v and consisting of vertices in A_1 . Then $r_2 l_{uv}$ is a path joining $r_2 u$ and $r_2 v$ which does not intersect S ,

since any vertex along $r_2 l_{uv}$ is in $r_2 A_1$, and $r_2 A_1 \cap S = \emptyset$. Therefore all the vertices in $r_2 A_1$ are in the same component B_1 of $G \setminus S$ such that $B_1 \cap A_1 = \emptyset$.

As in Assumption 1.1 (3), let $\partial_{A_1} S$ be the set consisting of all the vertices in S incident to a vertex in A_1 . Let $u \in A_1$, $w \in \partial_{A_1} S$, and e be the edge of G with endpoints u and w . Since $r_2 u \in B_1$, we have $r_2 w \in B_1 \cup \partial_{B_1} S$, since $r_2 u$ and $r_2 w$ are adjacent vertices in G . Let $l_{w,r_2 w}$ be a path in G joining w and $r_2 w$, starting from w and ending in $r_2 w$. Let p be the last vertex of S visited by $l_{w,r_2 w}$, and let q be the first vertex of $r_2 S$ visited by $l_{w,r_2 w}$. By the connectivity of S , there exists a path L_{wp} joining w and p and consisting of vertices of S ; also, there exists a path $L_{q,r_2 w}$ joining q and $r_2 w$ and consisting of vertices of $r_2 S$. By (4.4), $r_2 S \cap S = \emptyset$. Since $r_2 \in r_2 S$ and $r_2 \notin A_1$, the connectivity of $r_2 S$ implies that $r_2 S$ is in a component of $G \setminus S$ different from A_1 ; in particular $r_2 S \cap A_1 = \emptyset$. Moreover, since $r_2 w \in r_2 S$, $r_2 w \in B_1 \cup \partial_{B_1} S$, we have $r_2 S \subset B_1$. Let l_{pq} be the portion of $l_{w,r_2 w}$ between p and q . All the vertices along l_{pq} except p are outside S , hence they are in the same component of $G \setminus S$. Since $q \in r_2 S \subset B_1$, all the vertices along l_{pq} except p are in B_1 . Similarly, all the vertices along l_{pq} except q are outside $r_2 S$, hence they are in the same component of $G \setminus r_2 S$. Under the assumption that the length of the normal form of r_2 is strictly greater than the maximal length of normal forms of elements in $C_2(= S)$, we have $r_2 A_1 \cap S = \emptyset$. Since $p \in S$, $p \notin r_2 A_1$. Therefore no vertices in l_{pq} except p and q are in $A_1 \cup r_2 A_1$. It is straightforward to check that no vertices in $L_{wp} \cup L_{q,r_2 w}$ are in $A_1 \cup r_2 A_1$. Let $L_{w,r_2 w} = L_{wp} \cup l_{pq} \cup L_{q,r_2 w}$; then $L_{w,r_2 w}$ is a path joining w and $r_2 w$ in $G \setminus [A_1 \cup r_2 A_1]$. \square

4.2. Proof of Theorem 1.8 when Γ is an amalgamated free product. In this section, we prove Theorem 1.8 when Γ is an amalgamated free product. Let Γ be a finitely generated, infinite group which is a free product with amalgamation as described in (1). It suffices to construct a locally finite Cayley graph G of Γ such that SAWs on G have positive speed.

Choose a Cayley graph G_H (resp. G_K) for H (resp. K) such that any two vertices in C (resp. C') are joined by an edge. Let G be the graph obtained from the free product graph $G_H * G_K$ by gluing the vertices $u \in G_H * G_K$ and $w \in G_H * G_K$ satisfying the condition that there exists a vertex $v \in G_H * G_K$ such that $v^{-1}u = v^{-1}w \in C$.

Let

$$S' = S = C$$

It is not hard to check that for the locally finite Cayley graph G of Γ constructed above with the finite set of vertices S , Assumption 1.2 is satisfied and SAWs on G have positive speed.

4.3. HNN extension. In this section, we prove Theorems 1.7 and 1.8 when Γ is an HNN extension as described by (2).

Choose a set of representatives of the right cosets of C_1 in G , and a set of representatives of the right cosets of C_2 in G . We shall assume that 1 is the representatives of both C_1 and C_2 . The choice of coset representatives is to be fixed to the rest of the discussion.

dfn **Definition 4.6.** (Normal form for HNN extension [20]) Let Γ be the HNN extension with a presentation

$$\Gamma = \langle H, t | t^{-1}c_1t = \phi(c_1), \forall c_1 \in C_1 \rangle.$$

where H is a group, C_1, C_2 are two finite subgroups of H , and $\phi : C_1 \rightarrow C_2$ is a group isomorphism. A normal form is a sequence $g_0, t^{\epsilon_1}, \dots, t^{\epsilon_n}, g_n (n \geq 0)$ where

- g_0 is an arbitrary element of Γ ;
- if $\epsilon_i = -1$, then g_i is a representative of a coset of C_1 in Γ ;
- if $\epsilon_i = 1$, then g_i is a representative of a coset of C_2 in Γ ; and
- there is no consecutive subsequence $t^\epsilon, 1, t^{-\epsilon}$.

t47 **Theorem 4.7.** (Uniqueness of normal form [20]) Every element w of Γ with a presentation as in (2) has a unique representation as

$$w = g_0 t^{\epsilon_1} \cdot \dots \cdot t^{\epsilon_n} g_n,$$

where $g_0, t^{\epsilon_1}, \dots, t^{\epsilon_n}, g_n$ is a normal form. Let n be the **length** of the normal form of w .

4.3.1. Proof of Theorem 1.7 when Γ is an HNN extension. Let Γ be an infinite, finitely generated group, which is an HNN extension as described in (2). Let G_H be a locally finite Cayley graph for H with respect to a finite set of generators T_H satisfying $|T_H| < \infty$, $T_H = T_H^{-1}$ and $1 \notin T_H$.

Let G_0 be the Cayley graph of the HNN extension Γ with respect to the generator set $T_H \cup \{t, t^{-1}\}$. Let G be a locally finite Cayley graph of Γ with respect to a finite generator set T satisfying $T = T^{-1}$, $|T| < \infty$ and $1 \notin T$.

Let

$$\begin{aligned} D_0 &= \max_{v \in \Gamma, s \in T} \text{dist}_{G_0}(v, vs); \\ \text{s0} \quad (4.6) \quad S_0 &= \left\{ w \in \Gamma, \text{dist}_{G_0}(w, C_1 \cup C_2) \leq \frac{D_0}{2} \right\} \end{aligned}$$

Obviously C is finite since both C_1 and C_2 are finite.

css **Lemma 4.8.** Let $u, v \in \Gamma$ such that one of the followings hold

- (a) v has a normal form with $\epsilon_1 = 1$, $g_0 \in C_1$; and u has a normal form with $\epsilon_1 = -1$; or

- (b) v has a normal form with $\epsilon_1 = -1$, $g_0 \in C_2$; and u has a normal form with $\epsilon_1 = 1$; or
- (c) v has a normal form with $\epsilon_1 = 1$, $g_0 \in C_1$; and u has a normal form with $\epsilon_1 = 1$, $g_0 \in H \setminus C_1$; or
- (d) v has a normal form with $\epsilon_1 = -1$, $g_0 \in C_2$; and u has a normal form with $\epsilon_1 = -1$, $g_0 \in H \setminus C_2$;

Then any path in G_0 joining 1_Γ and $u^{-1}v$ must visit a vertex in $u^{-1}[C_1 \cup C_2]$

Proof. Let $l_{u^{-1}v}$ be a path in G_0 joining 1_Γ and $u^{-1}v$. Let y_1, \dots, y_n be all the vertices along $l_{u^{-1}v}$ such that one of the followings holds

- A. both edges incident to y_i along $l_{u^{-1}v}$ are t or t^{-1} ; or
- B. one edge incident to y_i along $l_{u^{-1}v}$ is t or t^{-1} ; the other edge incident to y_i along $l_{u^{-1}v}$ is an edge of G_H .

Count each vertex in Case A. twice in $\{y_1, \dots, y_n\}$, and count each vertex in Case B. once in $\{y_1, \dots, y_n\}$. We perform the following manipulations on the set $\{y_1, \dots, y_n\}$.

- (1) For all i 's satisfying $y_{i-1}^{-1}y_i = t^{-1}$, $y_i^{-1}y_{i+1} \in C_1$, and $y_{i+1}^{-1}y_{i+2} = t$, remove $y_{i-1}, y_i, y_{i+1}, y_{i+2}$ from $\{y_1, \dots, y_n\}$, and let $\{y_1^{(1)}, \dots, y_{n_1}^{(1)}\}$ be the new sequence;
- (2) For all i 's satisfying $[y_{i-1}^{(1)}]^{-1}y_i^{(1)} = t$, $[y_i^{(1)}]^{-1}y_{i+1}^{(1)} \in C_2$, and $[y_{i+1}^{(1)}]^{-1}y_{i+2}^{(1)} = t^{-1}$, remove $y_{i-1}^{(1)}, y_i^{(1)}, y_{i+1}^{(1)}, y_{i+2}^{(1)}$ from $\{y_1^{(1)}, \dots, y_{n_1}^{(1)}\}$, and let $\{y_1^{(2)}, \dots, y_{n_2}^{(2)}\}$ be the new sequence;

Assume we have obtained $\{y_1^{(2j)}, \dots, y_{n_{2j}}^{(2j)}\}$, then we perform the following inductive manipulations

- (1) For all i 's satisfying $[y_{i-1}^{(2j)}]^{-1}y_i^{(2j)} = t^{-1}$, $[y_i^{(2j)}]^{-1}y_{i+1}^{(2j)} \in C_1$, and $[y_{i+1}^{(2j)}]^{-1}y_{i+2}^{(2j)} = t$, remove $y_{i-1}^{(2j)}, y_i^{(2j)}, y_{i+1}^{(2j)}, y_{i+2}^{(2j)}$ from $\{y_1^{(2j)}, \dots, y_{n_{2j}}^{(2j)}\}$, and let $\{y_1^{(2j+1)}, \dots, y_{n_{2j+1}}^{(2j+1)}\}$ be the new sequence;
- (2) For all i 's satisfying $[y_{i-1}^{(2j+1)}]^{-1}y_i^{(2j+1)} = t$, $[y_i^{(2j+1)}]^{-1}y_{i+1}^{(2j+1)} \in C_2$, and $[y_{i+1}^{(2j+1)}]^{-1}y_{i+2}^{(2j+1)} = t^{-1}$, remove $y_{i-1}^{(2j+1)}, y_i^{(2j+1)}, y_{i+1}^{(2j+1)}, y_{i+2}^{(2j+1)}$ from $\{y_1^{(2j+1)}, \dots, y_{n_{2j+1}}^{(2j+1)}\}$, and let $\{y_1^{(2j+2)}, \dots, y_{n_{2j+2}}^{(2j+2)}\}$ be the new sequence.

We continue the above process until we obtain $\{y_1^{(2k)}, \dots, y_{n_{2k}}^{(2k)}\}$ such that

- (i) There are no i 's satisfying $[y_{i-1}^{(2k)}]^{-1}y_i^{(2k)} = t^{-1}$, $[y_i^{(2k)}]^{-1}y_{i+1}^{(2k)} \in C_1$, and $[y_{i+1}^{(2k)}]^{-1}y_{i+2}^{(2k)} = t$; and
- (ii) There are no i 's satisfying $[y_{i-1}^{(2k)}]^{-1}y_i^{(2k)} = t$, $[y_i^{(2k)}]^{-1}y_{i+1}^{(2k)} \in C_2$, and $[y_{i+1}^{(2k)}]^{-1}y_{i+2}^{(2k)} = t^{-1}$.

Obviously the above inductive process will terminate after finitely many steps. Then we have

$$y_1^{(2k)} = \eta_0; y_2^{(2k)} = \eta_1 t^{\xi_1}; \dots; y_{n_{2k}}^{(2k)} = \eta_0 t^{\xi_1} \dots t^{\xi_n}; u^{-1}v = y_{n_{2k}}^{(2k)} \eta_n.$$

where $\eta_i \in H$. Since $\{y_1^{(2k)}, \dots, y_{n_{2k}}^{(2k)}\} \subset \{y_1, \dots, y_n\}$, all the vertices in $\{y_1^{(2k)}, \dots, y_{n_{2k}}^{(2k)}\}$ are visited by $l_{u^{-1}v}$. Working from the right to the left of $\eta_0 t^{\xi_1} \dots t^{\xi_n} \eta_n$, we can change it to a normal form $\theta_0 t^{\xi_1} \dots t^{\xi_n} \theta_n$, such that $\theta_1, \dots, \theta_n$ are the chosen representatives for right cosets of C_1 or C_2 , depending on the sign of ξ_1, \dots, ξ_n , respectively, as explained in Definition 4.6.

Assume that u and v satisfy one of (a),(b),(c),(d). The normal form $\theta_0 t^{\xi_1} \dots t^{\xi_n} \theta_n$ of $u^{-1}v$ gives rise to a path by using a path τ_i ($0 \leq i \leq n$) in $\theta_0 t^{\xi_1} \dots t^{\xi_i} H$ to join $\theta_0 t^{\xi_1} \dots t^{\xi_i}$ and $\theta_0 t^{\xi_1} \dots t^{\xi_i} \eta_i$, and concatenating these path as well as the t -edge or t^{-1} -edge joining them. Then the path must visit a vertex in $u^{-1}[C_1 \cup C_2]$. It is straightforward to check that before working from the right to the left, one vertex in $\{y_1^{(2k)}, \dots, y_{n_{2k}}^{(2k)}\}$ is in $u^{-1}[C_1 \cup C_2]$. Since $l_{u^{-1}v}$ visits every vertex in $\{y_1^{(2k)}, \dots, y_{n_{2k}}^{(2k)}\}$, it must visit a vertex in $u^{-1}[C_1 \cup C_2]$. Then the proof is complete. \square

148 **Lemma 4.9.** *Let $G \setminus S_0$ be the subgraph obtained from G by removing all the vertices in S_0 as well as their incident edges. For any $u, v \in G \setminus S_0$ one of (a) (b) (c) (d) in Lemma 4.8 holds, u and v are in two distinct components of $G \setminus S_0$. Moreover, $G \setminus S_0$ has at least two infinite components.*

Proof. To show that u and v are in two distinct components of $G \setminus S_0$, it suffices to show that any path in G joining u and v must visit a vertex in S_0 .

Let l_{uv} be a path in G joining u and v . Assume that l_{uv} visits no vertices in S_0 . For any edge $e = \langle p, q \rangle \in l_{uv}$, p and q can be joined by a path in G_0 whose length does not exceed D_0 . Then there exists a path in G_0 joining u and v and visiting no vertices in $C_1 \cup C_2$ by the definition of S_0 in (4.6). This is equivalent to the condition that there exists a path in G_0 joining 1_Γ and $u^{-1}v$, which visits no vertices in $u^{-1}[C_1 \cup C_2]$. But this is a contradiction to Lemma 4.8.

For u and v satisfying the condition of the theorem, assume the lengths of the normal forms of u and v are strictly greater than the maximal lengths of normal forms of elements in S_0 . Assume the normal form of u (resp. v) has length n_1 (resp. n_2). Let $\epsilon_{n_1}(u)$ (resp. $\epsilon_{n_2}(v)$) be the exponent of the n_1 th (resp. n_2 th) t in the normal form of u (resp. v). Then $\{ut^{\epsilon_{n_1}(u)k}\}_{k=1}^\infty$ and $\{vt^{\epsilon_{n_2}(v)k}\}_{k=1}^\infty$ are in two distinct infinite components of $G \setminus S_0$. \square

Let S_1 be a finite, connected set of vertices in G containing S_0 . By Lemmas 4.1 and 4.9, the fact that $G \setminus S_0$ has at least two distinct infinite components implies that $G \setminus S$ has at least two distinct infinite components.

4.3.2. $C_1 = C_2 = H$. In this section, we consider the case when the group Γ is an HNN extension, as in Definition 4.6, such that $C_1 = C_2 = H$.

Lemma 4.10. *Assume that the group Γ is a finitely generated HNN extension, as in Definition 4.6, such that $C_1 = C_2 = H$. Then any locally finite Cayley graph G of Γ has two ends.*

Proof. It is a well-known fact that the number of ends of locally finite Cayley graphs of a finitely generated group do not depend on the choices of finite generating set. Therefore it suffices to prove the theorem for a specific choice of generating set. Let G be the Cayley graph with respect to the generating set $H \cup \{t, t^{-1}\}$. It is straightforward to check that G has two ends. \square

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4.3.3. $|C_1| = |C_2| < H$. Now we prove Theorem 1.7 when the group Γ is a finitely generated HNN extension, as in Definition 4.6, such that C_1 is a proper subset of H . Since C_1 is finite, this implies that $|C_1| = |C_2| < |H|$, C_2 is also a proper subset of H .

Lemma 4.11. *When the group Γ is a finitely generated HNN extension, as in Definition 4.6, such that C_1 is a proper subset of H , then there exists $a \in (0, 1]$, such that*

$$\limsup_{n \rightarrow \infty} |\{\pi_n : \|\pi_n\| \geq an\}|^{\frac{1}{n}} = \mu,$$

where π_n is an n -step SAW starting from 1_Γ , and μ is the connective constant.

Proof. It suffices to check Assumption 1.1, then the lemma follows from Theorem 1.3 A.

Let S be a finite set of vertices including S_0 such that S is connected.

Let $v \in S$ be incident to $w \in G \setminus S$. Let A_w be the component of $G \setminus S$ including w . Let k be an integer which is strictly greater than the maximal length of normal forms of elements in S .

- A. If w has a normal form with $g_0 \in H \setminus [C_2 \cup C_1]$, let $\phi(S, A_w) = t^{-2k}$;
- B. If w has a normal form with $g_0 \in C_1$, $\epsilon_1 = 1$, let $\phi(S, A_w) = t^{-2k}r_1$, where $r_1 \in H \setminus C_1$;
- C. If w has a normal form with $g_0 \in C_1 \setminus C_2$, $\epsilon_1 = -1$, and A_w contains no vertices satisfying Case A.; let $\phi(S, A_w) = t^{2k}$;
- D. If w has a normal form with $g_0 \in C_2$, $\epsilon_1 = -1$, let $\phi(S, A_w) = r_2 t^{-2k}$, where $r_2 \in H \setminus C_2$;
- E. If w has a normal form with $g_0 \in C_2 \setminus C_1$, $\epsilon_1 = 1$, and A_w contains no vertices satisfying Case A. or Case C.; let $\phi(S, A_w) = t^{-2k}$.

For Case A., $\phi(S, A_w)w$ and w are in distinct components of $G \setminus S$ by (b)(d) of Lemma 4.8. Let u be an arbitrary vertex in A_w , then the following cases might occur

- (i) u has a normal form with $g_0 \in C_1 \setminus C_2$, $\epsilon_1 = -1$;
- (ii) u has a normal form with $g_0 \in C_2 \setminus C_1$, $\epsilon_1 = 1$;

If $\phi(S, A_w)u \in S$, then $u \in t^{2k}S$, then u has normal form with $g_0 \in C_1$, $\epsilon_1 = 1$, but this is possible in neither Case (i) nor Case (ii). Therefore $\phi(S, A_w)A_w$ is in a component of $G \setminus S$ different from A_w .

For Case B. $\phi(S, A_w)w$ and w are in distinct components of $G \setminus S$ by (b) of Lemma 4.8. Note that any vertex in A_w has a normal form with $g_0 \in C_1$, $\epsilon_1 = 1$. Therefore $\phi(S, A_w)A_w$ is in a component of $G \setminus S$ different from A_w .

For Case C. $\phi(S, A_w)w$ and w are in distinct components of $G \setminus S$ by (a) of Lemma 4.8. Let u be an arbitrary vertex in A_w , then u may have a normal form with $g_0 \in C_2 \setminus C_1$, $\epsilon_1 = 1$. If $\phi(S, A_w)u \in S$, then $u \in t^{-2k}S$. But this is not possible since in this case the normal form of u satisfies $\epsilon_1 = -1$. Therefore $\phi(S, A_w)A_w$ is in a component of $G \setminus S$ different from A_w .

For Case D. $\phi(S, A_w)w$ and w are in distinct components of $G \setminus S$ by (d) of Lemma 4.8. Note that any vertex in A_w has a normal form with $g_0 \in C_2$, $\epsilon_1 = -1$. Therefore $\phi(S, A_w)A_w$ is in a component of $G \setminus S$ different from A_w .

For Case E. $\phi(S, A_w)w$ and w are in distinct components of $G \setminus S$ by (a) of Lemma 4.8. Note that any vertex in A_w has a normal form with $g_0 \in C_2 \setminus C_1$, $\epsilon_1 = 1$. Therefore $\phi(S, A_w)A_w$ is in a component of $G \setminus S$ different from A_w .

Let B_w be the component of $G \setminus S$ containing $\phi(S, A_w)A_w$. By the construction above, we have

$$\phi(S, A_w)S \cap S = \emptyset.$$

Therefore, $\phi(S, A_w)A_w \subset B_w$ implies

$$\boxed{\text{bw}} \quad (4.7) \quad \phi(S, A_w)S \subset B_w.$$

Also it is straight forward to check from Cases A.-E. that A_w and $\phi(S, A_w)A_w$ are in two distinct components of $G \setminus [\phi(S, A_w)S]$. For each $v \in \partial_{A_w}S$, we can construct a path l_v joining v and $\phi(S, A_w)v$ as follows

1. use a path l_1 in S to join v and a vertex u in $\partial_{B_w}S$ - this is possible by the connectivity of S ;
2. use a shortest path l_2 in B_w to join u and $\phi(S, A_w)S$; let x be the endpoint of l_2 ;
3. use a path l_3 in $\phi(S, A_w)S$ to join x and $\phi(S, A_w)v$

Let l_v be the concatenation of l_1 , l_2 , and l_3 .

We can check that $l_v \in G \setminus [A_w \cup \phi(S, A_w)A_w]$ as follows. The path $l_1 \subset S$, $S \cap A_w = \emptyset$ and $S \cap \phi(S, A_w)A_w = \emptyset$. Therefore $l_1 \in G \setminus [A_w \cup \phi(S, A_w)A_w]$. The path $l_3 \in \phi(S, A_w)S$; $\phi(S, A_w)S \cap \phi(S, A_w)A_w = \emptyset$ since $S \cap A_w = \emptyset$; $\phi(S, A_w)S \cap A_w = \emptyset$ by (4.7) and the fact that $B_w \cap A_w = \emptyset$. Therefore $l_3 \in G \setminus [A_w \cup \phi(S, A_w)A_w]$. The path $l_2 \in B_w$, hence $l_2 \cap A_w = \emptyset$. Since $[l_1 \cup l_2 \setminus \{x\}] \cap \phi(S, A_w)S = \emptyset$; and v is in

the same component of $G \setminus [\phi(S, A_w)S]$ as A_w , we obtain that $l_1 \cup l_2 \setminus \{x\}$ are in the same component of $G \setminus [\phi(S, A_w)S]$ as A_w and $[l_1 \cup l_2 \setminus \{x\}] \cap \phi(S, A_w)A_w = \emptyset$. Therefore $l_2 \in G \setminus [A_w \cup \phi(S, A_w)A_w]$.

The length of l_1 and l_3 are bounded above by $|S|$. We can make the length of l_2 to be bounded above by the distance of $\partial_{B_w}S$ and $\phi(S, A_w)S$, which is bounded above by the graph distance in G if 1_Γ and $\phi(S, A_w)1_\Gamma$. The latter is bounded by $2k + \max\{\text{dist}_G(1_\Gamma, r_1), \text{dist}_G(1_\Gamma, r_2)\}$. Hence if we choose

$$N = 2|S| + 2k + \max\{\text{dist}_G(1_\Gamma, r_1), \text{dist}_G(1_\Gamma, r_2)\},$$

then Assumption 1.1(3) is satisfied.

Therefore Theorem 1.7 in this Case follows from Theorem 1.3 A. \square

4.3.4. *Proof of Theorem 1.8 when Γ is an HNN extension.* Let Γ be an infinite, finitely generated graph which is an HNN extension as described by (2). It suffices to construct a locally finite Cayley graph G of Γ on which SAWs have positive speed.

First we consider the case when $C_1 = C_2 = H$. Let G be the Cayley graph of Γ with respect to generator set $H \cup \{t, t^{-1}\}$; i.e. any elements in H corresponds to an edge in G . Let $S' = S = H$. Note that $\phi \setminus S$ has two distinct infinite components. For any component A of $G \setminus S$, let $\phi(S, A)$ be the mapping from Γ to Γ changing each t in the normal form to t^{-1} and each t^{-1} in the normal form to t . Then Theorem 1.8 in this case follows from Theorem 1.3 B.

Now we consider the case when C_1 is a proper subset of H . Let T_H be a finite generator set of H satisfying $T_H = T_H^{-1}$, $1 \notin T_H$, $|T_H| < \infty$, and $[C_1 \cup C_2] \setminus \{1_\Gamma\} \subset T_H$. Let G be a Cayley graph of Γ with respect to the set of generators $T_H \cup \{t, t^{-1}\}$. Let S be defined as in Section 4.3.3, and let $S' = S$. Then Theorem 1.8 in this case follows from Theorem 1.3 B.

5. FREE PRODUCT GRAPH OF TWO QUASI-TRANSITIVE GRAPHS

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In this section, we prove Theorem 1.6.

Proof. Obviously G is an infinite, connected, quasi-transitive graph. Let $S = \{o\} \in V$. Then $G \setminus S$ has at least two infinite components. Indeed, Let $x, y \in V$ satisfy

$$\boxed{x} \quad (5.1) \quad x = x_1 \dots x_n;$$

$$\boxed{y} \quad (5.2) \quad y = y_1 \dots y_m;$$

where $m, n \geq 1$, $x_i, y_j \in V_1^\times \cup V_2^\times$, $I(x_i) \neq I(x_{i+1})$, $I(y_j) \neq I(y_{j+1})$ (see Definition 1.5 for notations). If $x_1 \in V_1^\times$ and $y_1 \in V_2^\times$, then x and y are in two distinct components of $G \setminus S$.

Let A (resp. B) be a component of $G \setminus S$, such that for any $x \in A$ (resp. $y \in B$), x (resp. y) has the form (5.1) (resp. (5.2)) with $x_1 \in V_1^\times$ (resp. $y_1 \in V_2^\times$). Let $u \in V_2^\times$ (resp. $w \in V_2^\times$), and define $\phi(S, A)x = ux$ (resp. $\phi(S, B)y = wy$). Then it

is straightforward to verify Assumption 1.2 with S chosen as above. Therefore the theorem follows from Theorem 1.8. \square

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REFERENCES

- [1] R. Bauerschmidt, H. Duminil-Copin, J. Goodman, and G. Slade, *Lectures on self-avoiding walks*, Probability and Statistical Physics in Two and More Dimensions (D. Ellwood, C. M. Newman, V. Sidoravicius, and W. Werner, eds.), Clay Mathematics Institute Proceedings, vol. 15, CMI/AMS publication, 2012, pp. 395–476.
- [2] I. Benjamini and O. Schramm, *Percolation beyond \mathbb{Z}^d : many questions and a few answers*, Electron. Commun. Probab. **1** (1996), 71–82.
- [3] D. Brydges and T. Spencer, *Self-avoiding walk in 5 or more dimensions*, Commun. Math. Phys. **97** (1985), 125–148.
- [4] H. Duminil-Copin and A. Hammond, *Self-avoiding walk is sub-ballistic*, Commun. Math. Phys. **324** (2013), 401–423.
- [5] H. Duminil-Copin and S. Smirnov, *The connective constant of the honeycomb lattice equals $\sqrt{2 + \sqrt{2}}$* , Ann. Math. **175** (2012), 1653–1665.
- [6] Z. Dvorak and B. Mohar, *Spectral radius of finite and infinite planar graphs and of graphs of bounded genus*, J. Combin. Theory Ser. B **100** (2010), 729–739.
- [7] P. Flory, *Principles of polymer chemistry*, Cornell University Press, 1953.
- [8] H. Frauenkron, M.S. Causo, and P. Grassberger, *Two-dimensional self-avoiding walks on a cylinder*, Phys. Rev. E **59** (1999), R16–R19.
- [9] L. Gilch and S. Muller, *Counting self-avoiding walks on free products of graphs*, Discrete Mathematics **340** (2017), 325–332.
- [10] G. Grimmett and Z. Li, *Cubic graphs and the golden mean*, <http://arxiv.org/abs/1610.00107>.
- [11] G. R. Grimmett and Z. Li, *Counting self-avoiding walks*, (2013), <http://arxiv.org/abs/1304.7216>.
- [12] ———, *Locality of connective constants*, (2014), <http://arxiv.org/abs/1412.0150>.
- [13] ———, *Strict inequalities for connective constants of regular graphs*, SIAM J. Disc. Math. **28** (2014), 1306–1333.
- [14] ———, *Bounds on connective constants of regular graphs*, Combinatorica **35** (2015), 279–294.
- [15] ———, *Connective constants and height functions of Cayley graphs*, (2015), <http://arxiv.org/abs/1501.00476>.
- [16] ———, *Self-avoiding walks and amenability*, (2015), <http://arxiv.org/abs/1510.08659>.
- [17] J. M. Hammersley, *Percolation processes II. The connective constant*, Proc. Camb. Phil. Soc. **53** (1957), 642–645.
- [18] T. Hara and G. Slade, *Self-avoiding walk in 5 or more dimensions. i. the critical behaviour*, Commun. Math. Phys. **147** (1992), 101–136.
- [19] H. Kesten, *On the number of self-avoiding walks*, J. Math. Phys. **4** (1963), 960–969.
- [20] R. Lyndon and P. Schupp, *Combinatorial group theory*, Springer-Verlag, 1977.
- [21] N. Madras and G. Slade, *The self-avoiding walk*, Birkhäuser, 1996.

- MW05 [22] N. Madras and C. Wu, *Self-avoiding walks on hyperbolic graphs*, *Combin. Probab. Comput.* **14** (2005), 523–548.
- BM88 [23] B. Mohar, *Isoperimetric inequalities, growth, and spectrum of graphs*, *Lin. Alg. Appl.* **103** (1988), 119–131.
- NP12 [24] A. Nachmias and Y. Peres, *Non-amenable Cayley graphs of high girth have $p_c < p_u$ and mean-field exponents*, *Electron. Commun. Probab.* **17** (2012), 1–8.
- PS00 [25] I. Pak and T. Smirnova-Nagnibeda, *On non-uniqueness of percolation on non-amenable cayley graphs*, *C.R.Acad.Sci.Paris* **33** (2000), 495–500.
- JS68 [26] J. Stallings, *On torsion-free groups with infinitely many ends*, *Ann. Math.* **88** (1968), 312–334.
- JS71 [27] ———, *Group theory and three-dimensional manifolds*, A James K. Whittemore Lecture in Mathematics given at Yale University, 1969, Yale Mathematical Monographs, vol. 4, Yale University Press, New Haven, Conn., 1971.

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